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HAMBLIN SMITH'S GEOMETRY:

BOOKS I AND II,

WITH

EXAMINATION PAPERS

BY

THOS. KIRKLAND, M.A.

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TORONTO
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ELEMENTS OF GEOMETRY

CONTAINING

BOOKS I. and II.

WITH

EXERCISES AND NOTES.

BY

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WITH

SELECTION OF EXAMINATION PAPERS, BY THOS. KIRKLAND, M.A., SCIENCE MASTER, NORMAL SCHOOL.

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PREFACE.

To preserve Euclid's order, to supply omissions, to remove defects, to give brief notes of explanation and simpler methods of proof in cases of acknowledged difficulty—such are the main objects of this Edition of the Elements.

The work is based on the Greek text, as it is given in the Editions of August and Peyrard. To the suggestions of the late Professor De Morgan, published in the Companion to the British Almanack for 1849, I have paid constant deference.

A limited use of symbolic representation, wherein the symbols stand for words and not for operations, is generally regarded as desirable, and I have been assured, by the highest authorities on this point, that the symbols employed in this book are admissible in the Examinations at Oxford and Cambridge.[1]

I have generally followed Euclid's method of proof, but not to the exclusion of other methods recom-

[1] I regard this point as completely settled in Cambridge by the following notices prefixed to the papers on Euclid set in the Senate-House Examinations:

I. In the Previous Examination:

In answers to these questions any intelligible symbols and abbreviations may be used.

II. In the Mathematical Tripos:

In answers to the questions on Euclid the symbol — must not be used. The only abbreviation admitted for the square on AB *is "sq. on* AB,*" and for the rectangle contained by* AB *and* CD, "*rect.* AB, CD."

mended by their simplicity, such as the demonstrations by which I propose to replace (at least for a first reading) the difficult Theorems 5 and 7 in the First Book. I have also attempted to render many of the proofs, as for instance Propositions 2, 13, and 35 in Book I., and Proposition 13 in Book II., less confusing to the learner.

In Propositions 4, 5, 6, 7, and 8 of the Second Book I have ventured to make an important change in Euclid's mode of exposition, by omitting the diagonals from the diagrams and the gnomons from the text.

In the Third Book I have deviated with even greater boldness from the precise line of Euclid's method. For it is in treating of the properties of the circle that the importance of certain matters, to which reference is made in the Notes of the present volume, is fully brought out. I allude especially to the application of Superposition as a test of equality, to the conception of an Angle as a magnitude capable of unlimited increase, and to the development of the methods connected with Loci and Symmetry.

The Exercises have been selected with considerable care, chiefly from the Senate House Examination Papers. They are intended to be progressive and easy, so that a learner may from the first be induced to work out something for himself.

I desire to express my thanks to the friends who have improved this work by their suggestions, and to beg for further help of the same kind.

J. HAMBLIN SMITH.

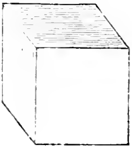
CAMBRIDGE, 1873.

ELEMENTS OF GEOMETRY.

INTRODUCTORY REMARKS.

WHEN a block of stone is hewn from the rock, we call it a Solid *Body*. The stone-cutter shapes it, and brings it into that which we call *regularity of form;* and then it becomes a Solid *Figure*.

Now suppose the figure to be such that the block has six flat sides, each the exact counterpart of the others; so that, to one who stands facing a corner of the block, the three sides which are visible present the appearance represented in this diagram.



Each side of the figure is called a *Surface;* and when smoothed and polished, it is called a *Plane* Surface.

The sharp and well-defined edges, in which each pair of sides meets, are called *Lines*.

The place, at which any three of the edges meet, is called a *Point*.

A *Magnitude* is anything which is made up of parts in any way like itself. Thus, a line is a magnitude; because we may regard it as made up of parts which are themselves lines.

The properties Length, Breadth (or Width), and Thickness (or Depth or Height) of a body are called its *Dimensions*.

We make the following distinction between Solids, Surfaces, Lines, and Points:

A Solid has three dimensions, Length, Breadth, Thickness.

A Surface has two dimensions, Length, Breadth.

A Line has one dimension, Length.

A point has no dimensions.

BOOK I.

DEFINITIONS.

I. A POINT is that which has no parts.

This is equivalent to saying that a Point has no magnitude, since we define it as that which cannot be divided into smaller parts.

II. A LINE is length without breadth.

We cannot conceive a visible line without breadth; but we can reason about lines as if they had no breadth, and this is what Euclid requires us to do.

III. The EXTREMITIES of finite LINES are points.

A point marks *position*, as for instance, the place where a line begins or ends, or meets or crosses another line.

IV. A STRAIGHT LINE is one which lies in the same direction from point to point throughout its length.

V. A SURFACE is that which has length and breadth only.

VI. The EXTREMITIES of a SURFACE are lines.

VII. A PLANE SURFACE is one in which, if any two points be taken, the straight line between them lies wholly in that surface.

Thus the ends of an uncut cedar-pencil are plane surfaces; but the rest of the surface of the pencil is not a plane surface, since two points may be taken in it such that the *straight* line joining them will not lie on the surface of the pencil.

In our introductory remarks we gave examples of a Surface, a Line, and a Point, as we know them through the evidence of the senses.

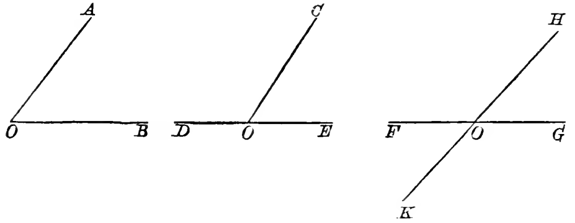
The Surfaces, Lines, and Points of Geometry may be regarded as mental pictures of the surfaces, lines, and points which we know from experience.

It is, however, to be observed that Geometry requires us to conceive the possibility of the existence

of a Surface apart from a Solid body,
of a Line apart from a Surface.
of a Point apart from a Line.

VIII. When two straight lines meet one another, the inclination of the lines to one another is called an ANGLE.

When *two* straight lines have one point common to both, they are said to *form* an angle (or angles) at that point. The point is called the *vertex* of the angle (or angles), and the lines are called the *arms* of the angle (or angles).

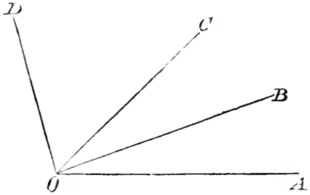


Thus, if the lines OA, OB are terminated at the same point O, they form an angle, which is called *the angle at* O, or *the angle* AOB, or *the angle* BOA,—the letter which marks the vertex being put between those that mark the arms.

Again, if the line CO meets the line DE at a point in the line DE, so that O is a point common to both lines, CO is said to make with DE the angles COD, COE; and these (as having one arm, CO, common to both) are called *adjacent* angles.

Lastly, if the lines FG, HK cut each other in the point O, the lines make with each other four angles FOH, HOG, GOK, KOF; and of these GOH, FOK are called *vertically opposite* angles, as also are FOH and GOK.

When *three or more* straight lines as OA, OB, OC, OD have a point O common to all, the angle formed by one of them, OD,



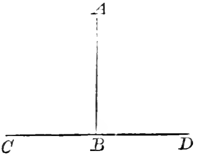
with OA may be regarded as being made up of the angles AOB, BOC, COD; that is, we may speak of the angle AOD as a whole, of which the parts are the angles AOB, BOC, and COD.

Hence we may regard an angle as a *Magnitude*, inasmuch as any angle may be regarded as being made up of parts which are themselves angles.

The size of an angle depends in no way on the length of the arms by which it is bounded.

We shall explain hereafter the restriction on the magnitude of angles enforced by Euclid's definition, and the important results that follow an extension of the definition.

IX. When a straight line (as AB) meeting another straight line (as CD) makes the adjacent angles (ABC and ABD) equal to one another, each of the angles is called a RIGHT ANGLE; and each line is said to be a PERPENDICULAR to the other.



X. An OBTUSE ANGLE is one which is greater than a right angle.

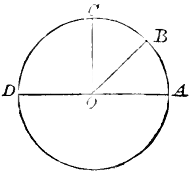
XI. An ACUTE ANGLE is one which is less than a right angle.

XII. A FIGURE is that which is enclosed by one or more boundaries.

XIII. A CIRCLE is a plane figure contained by one line, which is called the CIRCUMFERENCE, and is such, that all straight lines drawn to the circumference from a certain point (called the CENTRE) within the figure are equal to one another.

XIV. Any straight line drawn from the centre of a circle to the circumference is called a RADIUS.

XV. A DIAMETER of a circle is a straight line drawn through the centre and terminated both ways by the circumference.



Thus, in the diagram, O is the centre of the circle $ABCD$, OA, OB, OC, OD are Radii of the circle, and the straight line AOD is a Diameter. Hence the radius of a circle is half the diameter.

XVI. A SEMICIRCLE is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XVII. RECTILINEAR figures are those which are contained by straight lines.

The PERIMETER (or Periphery) of a rectilinear figure is the sum of its sides.

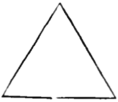
XVIII. A TRIANGLE is a plane figure contained by three straight lines.

XIX. A QUADRILATERAL is a plane figure contained by four straight lines.

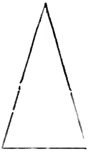
XX. A POLYGON is a plane figure contained by more than four straight lines.

When a polygon has all its sides equal and all its angles equal it is called a *regular* polygon.

XXI. An Equilateral Triangle is one which has all its sides equal.



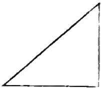
XXII. An Isosceles Triangle is one which has two sides equal.



The third side is often called the *base* of the triangle.

The term *base* is applied to any one of the sides of a triangle to distinguish it from the other two, especially when they have been previously mentioned.

XXIII. A Right-angled Triangle is one in which one of the angles is a right angle.



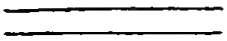
The side *subtending*, that is, *which is opposite* the right angle, is called the *Hypotenuse*.

XXIV. An Obtuse-angled Triangle is one in which one of the angles is obtuse.

It will be shewn hereafter that a triangle can have only one of its angles either equal to, or greater than, a right angle.

XXV. An Acute-angled Triangle is one in which all the angles are acute.

XXVI. Parallel Straight Lines are such as, being in the same plane, never meet when continually produced in both directions.



Euclid proceeds to put forward Six Postulates, or Requests, that he may be allowed to make certain assumptions on the construction of figures and the properties of geometrical magnitudes.

Postulates

Let it be granted—

I. That a straight line may be drawn from any one point to any other point.

II. That a terminated straight line may be produced to any length in a straight line.

III. That a circle may be described from any centre at any distance from that centre.

IV. That all right angles are equal to one another.

V. That two straight lines cannot enclose a space.

VI. That if a straight line meet two other straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced shall at length meet upon that side, on which are the angles, which are together less than two right angles.

The word rendered "Postulates" is in the original *αἰτήματα*, "requests."

In the first three Postulates Euclid states the use, under certain restrictions, which he desires to make of certain instruments for the construction of lines and circles.

In Post. I. and II. he asks for the use of the straight ruler, wherewith to draw straight lines. The restriction is, that the ruler is not supposed to be marked with divisions so as to measure lines.

In Post. III. he asks for the use of a pair of compasses, wherewith to describe a circle, whose centre is at one extremity of a given line, and whose circumference passes through the other extremity of that line. The restriction is, that the compasses are not supposed to be capable of conveying distances.

Post. IV. and V. refer to simple geometrical facts, which Euclid desires to take for granted.

Post. VI. may, as we shall shew hereafter, be deduced from a more simple Postulate. The student must defer the consideration of this Postulate, till he has reached the 17th Proposition of Book I.

Euclid next enumerates, as statements of fact, nine Axioms

or, as he calls them, Common Notions, applicable (with the exception of the eighth) to all kinds of magnitudes, and not necessarily restricted, as are the Postulates, to *geometrical* magnitudes.

AXIOMS.

I. Things which are equal to the same thing are equal to one another.

II. If equals be added to equals, the wholes are equal.

III. If equals be taken from equals, the remainders are equal.

IV. If equals and unequals be added together, the wholes are unequal.

V. If equals be taken from unequals, or unequals from equals, the remainders are unequal.

VI. Things which are double of the same thing, or of equal things, are equal to one another.

VII. Things which are halves of the same thing, or of equal things, are equal to one another.

VIII. Magnitudes which coincide with one another are equal to one another.

IX. The whole is greater than its part.

With his Common Notions Euclid takes the ground of authority, saying in effect, "To my Postulates I request, to my Common Notions I claim, your assent."

Euclid develops the science of Geometry in a series of Propositions, some of which are called Theorems and the rest Problems, though Euclid himself makes no such distinction.

By the name *Theorem* we understand a truth, capable of demonstration or proof by deduction from truths previously admitted or proved.

By the name *Problem* we understand a construction, capable of being effected by the employment of principles of construction previously admitted or proved.

A *Corollary* is a Theorem or Problem easily deduced from, or effected by means of, a Proposition to which it is attached.

We shall divide the First Book of the Elements into three sections. The reason for this division will appear in the course of the work.

SYMBOLS AND ABBREVIATIONS USED IN BOOK I.

∵	*for*	because
∴	....	therefore
=	....	is (or are) equal to
∠	....	angle
△	....	triangle
⊙	*for*	circle
○ce	....	circumference
∥	....	parallel
▱	....	parallelogram
⊥	....	perpendicular

equilat.	.....	equilateral
extr.	....	exterior
intr.	....	interior
pt	....	point
rectil.	......	rectilinear
reqd.	....	required
rt	....	right
sq.	....	square
sqq.	....	squares
st	....	straight

It is well known that one of the chief difficulties with learners of Euclid is to distinguish between what is assumed, or given, and what has to be proved in some of the Propositions. To make the distinction clearer we shall put in italics the statements of what has to be done in a Problem, and what has to be proved in a Theorem. The last line in the proof of every Proposition states, that what had to be done or proved has been done or proved.

The letters Q. E. F. at the end of a Problem stand for *Quod erat faciendum.*

The letters Q. E. D. at the end of a Theorem stand for *Quod erat demonstrandum.*

In the marginal references :

Post. stands for Postulate.
Def. .... Definition.
Ax. .... Axiom.
I. 1. .... Book I. Proposition 1.

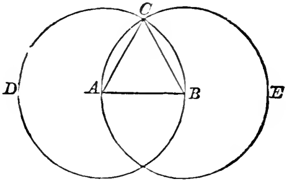
Hyp. stands for Hypothesis, *supposition*, and refers to something granted, or assumed to be true.

SECTION I.

On the Properties of Triangles.

Proposition I. Problem.

To describe an equilateral triangle on a given straight line.



Let AB be the given st. line.

It is required to describe an equilat. $\triangle$ *on* AB.

With centre A and distance AB describe $\odot$ BCD. Post. 3.
With centre B and distance BA describe $\odot$ ACE. Post. 3.

From the pt. C, in which the $\odot$s cut one another, draw the st. lines CA, CB. Post. 1.

Then will ABC be an equilat. $\triangle$.

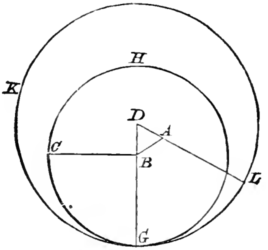
For	$\because$ A is the centre of $\odot$ BCD,	
	$\therefore$ $AC=AB$.	Def. 13
And	$\because$ B is the centre of $\odot$ ACE,	
	$\therefore$ $BC=AB$.	Def. 13.
Now	$\because$ AC, BC are each$=AB$,	
	$\therefore$ $AC=BC$.	Ax. 1.

Thus AC, AB, BC are all equal, and an equilat. $\triangle$ ABC has been described on AB.

Q. E. F.

Proposition II. Problem.

From a given point to draw a straight line equal to a given straight line.



Let A be the given pt., and BC the given st. line.

It is required to draw from A a st. line equal to BC.

From A to B draw the st. line AB. Post. 1.

On AB describe the equilat. $\triangle$ ABD. I. 1.

With centre B and distance BC describe $\odot$ CGH. Post. 3.

Produce DB to meet the $\bigcirc$ce CGH in G.

With centre D and distance DG describe $\odot$ GKL. Post. 3.

Produce DA to meet the $\bigcirc$ce GKL in L.

Then will $AL=BC$.

For $\because$ B is the centre of $\odot$ CGH,

$\therefore$ $BC=BG$. Def. 13.

And $\because$ D is the centre of $\odot$ GKL,

$\therefore$ $DL=DG$. Def. 13.

And parts of these, DA and DB, are equal. Def. 21.

$\therefore$ remainder AL=remainder BG. Ax. 3.

But $BC=BG$;

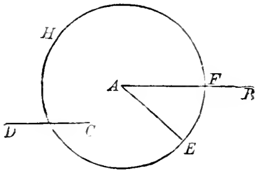
$\therefore$ $AL=BC$. Ax. 1.

Thus from pt. A a st. line AL has been drawn$=BC$.

Q. E. F.

Proposition III. Problem.

From the greater of two given straight lines to cut off a part equal to the less.



Let AB be the greater of the two given st. lines AB, CD.

It is required to cut off from AB a part $= CD$.

From A draw the st. line $AE = CD$. I. 2.

With centre A and distance AE describe ⊙ EFH, cutting AB in F.

Then will $AF = CD$.

For $\because$ A is the centre of ⊙ EFH,

$\therefore$ $AF = AE$.

But $AE = CD$;

$\therefore$ $AF = CD$. Ax. 1.

Thus from AB a part AF has been cut off $= CD$.

Q. E. F.

Exercises.

1. Shew that if straight lines be drawn from A and B in the diagram of Prop. I. to the other point in which the circles intersect, another equilateral triangle will be described on AB.

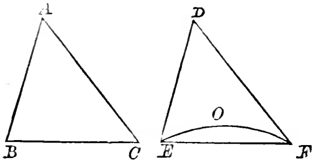
2. By a construction similar to that in Prop. III. produce the less of two given straight lines that it may be equal to the greater.

3. Draw a figure for the case in Prop. II., in which the given point coincides with B.

4. By a similar construction to that in Prop. I. describe on a given straight line an isosceles triangle, whose equal sides shall be each equal to another given straight line.

Proposition IV. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another, they must have their third sides equal; and the two triangles must be equal, and the other angles must be equal, each to each, viz. those to which the equal sides are opposite.



In the $\triangle$s ABC, DEF,

let $AB=DE$, and $AC=DF$, and $\angle\ BAC=\angle\ EDF$.

Then must $BC=EF$ *and* $\triangle\ ABC=\triangle\ DEF$, *and the other* $\angle$ *s, to which the equal sides are opposite, must be equal, that is,* $\angle\ ABC=\angle\ DEF$ *and* $\angle\ ACB=\angle\ DFE$.

For, if $\triangle\ ABC$ be applied to $\triangle\ DEF$,

so that A coincides with D, and AB falls on DE,

then $\because AB=DE$, $\therefore B$ will coincide with E.

And $\because AB$ coincides with DE, and $\angle\ BAC=\angle\ EDF$, Hyp.

$\therefore AC$ will fall on DF.

Then $\because AC=DF$, $\therefore C$ will coincide with F.

And $\because B$ will coincide with E, and C with F,

$\therefore BC$ will coincide with EF;

for if not, let it fall otherwise as EOF: then the two st. lines BC, EF will enclose a space, which is impossible. Post. 5.

$\therefore BC$ will coincide with and $\therefore$ is equal to EF, Ax. 8.

and $\triangle\ ABC$.. $\triangle\ DEF$,

and $\angle\ ABC$.. $\angle\ DEF$,

and $\angle\ ACB$.. $\angle\ DFE$.

Q. E. D.

NOTE 1. *On the Method of Superposition.*

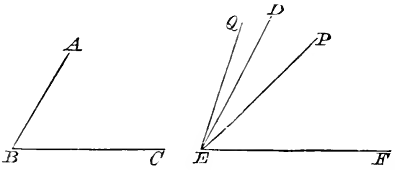
Two geometrical magnitudes are said, in accordance with Ax. VIII. to be *equal*, when they can be so placed that the boundaries of the one coincide with the boundaries of the other.

Thus, two straight lines are equal, if they can be so placed that the points at their extremities coincide: and two angles are equal, if they can be so placed that their vertices coincide in position and their arms in direction: and two triangles are equal, if they can be so placed that their sides coincide in direction and magnitude.

In the application of the test of equality by this *Method of Superposition*, we assume that an angle or a triangle may be moved from one place, turned over, and put down in another place, without altering the relative positions of its boundaries.

We also assume that if one part of a straight line coincide with one part of another straight line, the other parts of the lines also coincide in direction; or, that straight lines, which coincide in two points, coincide when produced.

The method of Superposition enables us also to compare magnitudes of the same kind that are unequal. For example, suppose ABC and DEF to be two given angles.



Suppose the arm BC to be placed on the arm EF, and the vertex B on the vertex E.

Then, if the arm BA coincide in direction with the arm ED, the angle ABC is equal to DEF.

If BA fall between ED and EF in the direction EP, ABC is less than DEF.

If BA fall in the direction EQ so that ED is between EQ and EF, ABC is greater than DEF.

Note 2. *On the Conditions of Equality of two Triangles.*

A Triangle is composed of six parts, three sides and three angles.

When the six parts of one triangle are equal to the six parts of another triangle, each to each, the Triangles are said to be equal in all respects.

There are four cases in which Euclid proves that two triangles are equal in all respects; viz., when the following parts are equal in the two triangles.

1. Two sides and the angle between them. I. 4.
2. Two angles and the side between them. I. 26.
3. The three sides of each. I. 8.
4. Two angles and the side opposite one of them. I. 26.

The Propositions, in which these cases are proved, are the most important in our First Section.

The first case we have proved in Prop. IV.

Availing ourselves of the method of superposition, we can prove Cases 2 and 3 by a process more simple than that employed by Euclid, and with the further advantage of bringing them into closer connexion with Case 1. We shall therefore give three Propositions, which we designate A, B, and C, in the Place of Euclid's Props. V. VI. VII. VIII.

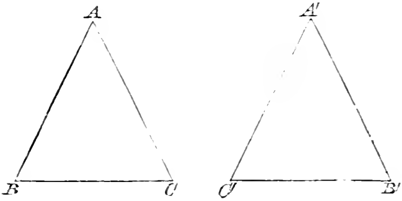
The displaced Propositions will be found on pp. 108-112.

Proposition A corresponds with Euclid I. 5.
.... B .... I. 26, first part.
.... C .... I. 8.

Proposition A. Theorem.

If two sides of a triangle be equal, the angles opposite those sides must also be equal.

Fig. 1. Fig. 2.



In the isosceles triangle ABC, let $AC=AB$. (Fig. 1.)

Then must $\angle ABC = \angle ACB$

Imagine the $\triangle ABC$ to be taken up, turned round, and set down again in a reversed position as in Fig. 2, and designate the angular points A', B', C'.

Then in $\triangle$s ABC, $A'C'B'$,

$\because$ $AB=A'C'$, and $AC=A'B'$, and $\angle BAC = \angle C'A'B'$,

$\therefore \angle ABC = \angle A'C'B'$. I. 4.

But $\angle A'C'B' = \angle ACB$;

$\therefore \angle ABC = \angle ACB$. Ax. 1.

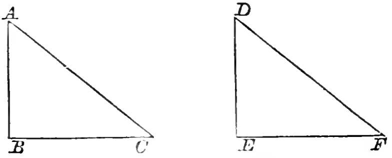
Q.E.D.

Cor. Hence every equilateral triangle is also equiangular.

Note. When one side of a triangle is distinguished from the other sides by being called the *Base*, the angular point opposite to that side is called the *Vertex* of the triangle.

Proposition B. Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and the sides adjacent to the equal angles in each also equal; then must the triangles be equal in all respects.



In $\triangle$s ABC, DEF,

let $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$, and $BC = EF$.

Then must $AB = DE$, *and* $AC = DF$, *and* $\angle BAC = \angle EDF$.

For if $\triangle DEF$ be applied to $\triangle ABC$, so that E coincides with B, and EF falls on BC;

then $\because EF = BC$, $\therefore F$ will coincide with C;

and $\because \angle DEF = \angle ABC$, $\therefore ED$ will fall on BA;

$\therefore D$ will fall on BA or BA produced.

Again, $\because \angle DFE = \angle ACB$, $\therefore FD$ will fall on CA;

$\therefore D$ will fall on CA or CA produced.

$\therefore D$ must coincide with A, the only pt. common to BA and CA.

$\therefore DE$ will coincide with and $\therefore$ is equal to AB,

and DF .... AC,

and $\angle EDF$ .... $\angle BAC$,

and $\triangle DEF$ .... $\triangle ABC$;

and $\therefore$ the triangles are equal in all respects.

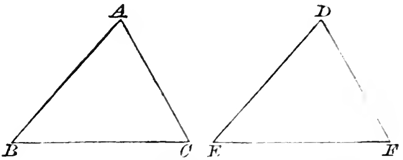
Q. E. D.

Cor. Hence, by a process like that in Prop. A, we can prove the following theorem:

If two angles of a triangle be equal the sides which subtend them are also equal (Eucl. I. 6.)

Proposition C. Theorem.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles must be equal in all respects.

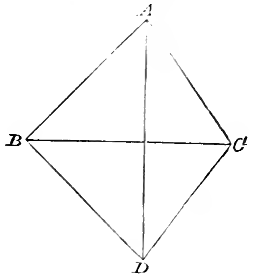


Let the three sides of the $\triangle$s ABC, DEF be equal, each to each, that is, $AB = DE$, $AC = DF$, and $BC = EF$.

Then must the triangles be equal in all respects.

Imagine the $\triangle DEF$ to be turned over and applied to the $\triangle ABC$, in such a way that EF coincides with BC, and the vertex D falls on the side of BC opposite to the side on which A falls; and join AD.

Case I. When AD passes through BC.



Then in $\triangle ABD$, $\because BD = BA$, $\therefore \angle BAD = \angle BDA$, I. A.

And in $\triangle ACD$, $\because CD = CA$, $\therefore \angle CAD = \angle CDA$, I. A.

$\therefore$ sum of $\angle$s BAD, CAD = sum of $\angle$s BDA, CDA, Ax. 2.

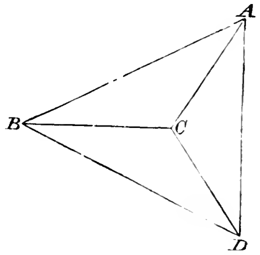
that is, $\angle BAC = \angle BDC$.

Hence we see, referring to the original triangles, that

$\angle BAC = \angle EDF$.

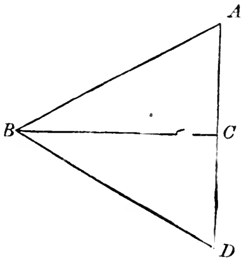
$\therefore$, by Prop. 4, the triangles are equal in all respects.

Case II. When the line joining the vertices does not pass through BC.



Then in $\triangle ABD$, $\because BD=BA$, $\therefore \angle BAD = \angle BDA$, I. A.
And in $\triangle ACD$, $\because CD=CA$, $\therefore \angle CAD = \angle CDA$, I. A.
Hence since the whole angles BAD, BDA are equal.
and parts of these CAD, CDA are equal.
$\therefore$ the remainders BAC, BDC are equal. Ax. 3.
Then, as in Case I., the equality of the original triangles may be proved.

Case III. When AC and CD are in the same straight line.



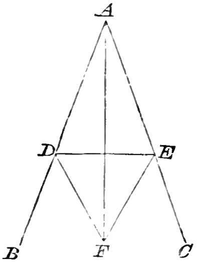
Then in $\triangle ABD$, $\because BD=BA$, $\therefore \angle BAD = \angle BDA$, I. A.
that is, $\angle BAC = \angle BDC$.

Then, as in Case I., the equality of the original triangles may be proved.

Q. E. D.

Proposition IX. Problem.

To bisect a given angle.



Let BAC be the given angle.

It is required to bisect $\angle BAC$.

In AB take any pt. D.

In AC make $AE=AD$, and join DE.

On DE, on the side remote from A, describe an equilat. $\triangle DFE$. I. 1.

Join AF. Then AF will bisect $\angle BAC$.

For in $\triangle$s AFD, AFE,

$\because AD=AE$, and AF is common, and $FD=FE$,

$\therefore \angle DAF = \angle EAF$, I. c.

that is, $\angle BAC$ is bisected by AF.

Q. E. F.

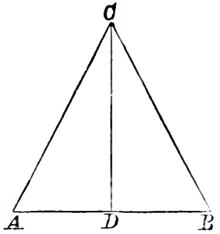
Ex. 1. Shew that we can prove this Proposition by means of Prop. IV. and Prop. A., without applying Prop. C.

Ex. 2. If the equilateral triangle, employed in the construction, be described with its vertex towards the given angle; shew that there is one case in which the construction will fail, and two in which it will hold good.

Note.—The line dividing an angle into two equal parts is called the Bisector of the angle.

Proposition X. Problem.

To bisect a given finite straight line.



Let AB be the given st. line.

It is required to bisect AB.

On AB describe an equilat. $\triangle ACB$. I. 1.

Bisect $\angle ACB$ by the st. line CD meeting AB in D; I. 9.

then AB shall be bisected in D.

For in $\triangle$s ACD, BCD,

$\because AC=BC$, and CD is common, and $\angle ACD = \angle BCD$,

$\therefore AD=BD$; I. 4.

$\therefore AB$ is bisected in D.

Q. E. F.

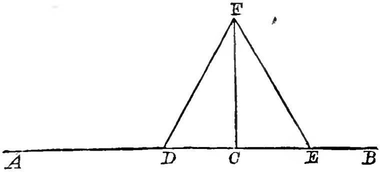
Ex. 1. The straight line, drawn to bisect the vertical angle of an isosceles triangle, also bisects the base.

Ex. 2. The straight line, drawn from the vertex of an isosceles triangle to bisect the base, also bisects the vertical angle.

Ex. 3. Produce a given finite straight line to a point, such that the part produced may be one-third of the line, which is made up of the whole and the part produced.

Proposition XI. Problem.

To draw a straight line at right angles to a given straight line from a given point in the same.



Let AB be the given st. line, and C a given pt. in it.

It is required to draw from C a st. line $\perp$ to AB.

Take any pt. D in AC, and in CB make $CE=CD$.

On DE describe an equilat. $\triangle$ DFE. I. 1.

Join FC. FC shall be $\perp$ to AB.

For in $\triangle$s DCF, ECF,

$\because$ $DC=CE$, and CF is common, and $FD=FE$,

$\therefore$ $\angle DCF = \angle ECF$; I. c.

and $\therefore$ FC is $\perp$ to AB. Def. 9.

Q. E. F.

Cor. To draw a straight line at right angles to a given straight line AC from one extremity, C, take any point D in AC, produce AC to E, making $CE=CD$, and proceed as in the proposition.

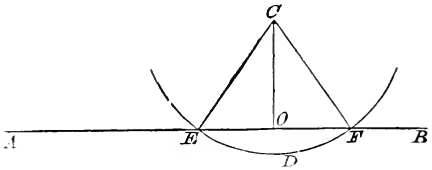
Ex. 1. Shew that in the diagram of Prop. ix. AF and ED intersect each other at right angles, and that ED is bisected by AF.

Ex. 2. If O be the point in which two lines, bisecting AB and AC, two sides of an equilateral triangle, at right angles, meet; shew that OA, OB, OC are all equal.

Ex. 3. Shew that Prop. xi. is a particular case of Prop. ix.

Proposition XII. Problem.

To draw a straight line perpendicular to a given straight line of an unlimited length from a given point without it.



Let AB be the given st. line of unlimited length; C the given pt. without it.

It is required to draw from C a st. line $\perp$ to AB.

Take any pt. D on the other side of AB.

With centre C and distance CD describe a ⊙ cutting AB in E and F.

Bisect EF in O, and join CE, CO, CF. I. 10

Then CO shall be $\perp$ to AB.

For in $\triangle$s COE, COF,

$\because$ $EO=FO$, and CO is common, and $CE=CF$,

$\therefore$ $\angle COE = \angle COF$; I. c.

$\therefore$ CO is $\perp$ to AB. Def. 9.

Q. E. F.

Ex. 1. If the straight line were not of unlimited length, how might the construction fail?

Ex. 2. If in a triangle the perpendicular from the vertex on the base bisect the base, the triangle is isosceles.

Ex. 3. The lines drawn from the angular points of an equilateral triangle to the middle points of the opposite sides are equal.

Miscellaneous Exercises on Props. I. to XII.

1. Draw a figure for Prop. II. for the case when the given point A is

 (α) below the line BC and to the right of it.
 (β) below the line BC and to the left of it.

2. Divide a given angle into four equal parts.

3. The angles B, C, at the base of an isosceles triangle, are bisected by the straight lines BD, CD, meeting in D; shew that BDC is an isosceles triangle.

4. D, E, F are points taken in the sides BC, CA, AB, of an equilateral triangle, so that $BD=CE=AF$. Shew that the triangle DEF is equilateral.

5. In a given straight line find a point equidistant from two given points; 1st, on the same side of it; 2d, on opposite sides of it.

6. ABC is a triangle having the angle ABC acute. In BA, or BA produced, find a point D such that $BD=CD$.

7. The equal sides AB, AC, of an isosceles triangle ABC are produced to points F and G, so that $AF=AG$. BG and CF are joined, and H is the point of their intersection. Prove that $BH=CH$, and also that the angle at A is bisected by AH.

8. BAC, BDC are isosceles triangles, standing on opposite sides of the same base BC. Prove that the straight line from A to D bisects BC at right angles.

9. In how many directions may the line AE be drawn in Prop. III.?

10. The two sides of a triangle being produced, if the angles on the other side of the base be equal, shew that the triangle is isosceles.

11. ABC, ABD are two triangles on the same base AB and on the same side of it, the vertex of each triangle being outside the other. If $AC=AD$, shew that BC cannot $=BD$.

12. From C any point in a straight line AB, CD is drawn at right angles to AB, meeting a circle described with centre A and distance AB in D: and from AD, AE is cut off $=AC$: shew that AEB is a right angle.

Proposition XIII. Theorem.

The angles which one straight line makes with another upon one side of it are either two right angles, or together equal to two right angles.

Fig. 1.

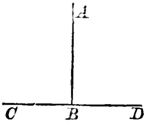
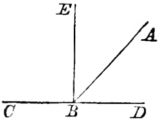


Fig. 2.



Let AB make with CD upon one side of it the $\angle$ s ABC, ABD.

Then must these be either two rt. $\angle$ *s,*
or together equal to two rt. $\angle$ *s*

First, if $\angle ABC = \angle ABD$ as in Fig. 1,
each of them is a rt. $\angle$. Def. 9.

Secondly, if $\angle ABC$ be not $= \angle ABD$, as in Fig. 2,
from B draw $BE \perp$ to CD. I. 11.

Then sum of $\angle$ s ABC, ABD = sum of $\angle$ s EBC, EBA, ABD,
and sum of $\angle$ s EBC, EBD = sum of $\angle$ s EBC, EBA, ABD;
$\therefore$ sum of $\angle$ s ABC, ABD = sum of $\angle$ s EBC, EBD;
Ax. 1.

$\therefore$ sum of $\angle$ s ABC, ABD = sum of a rt. $\angle$ and a rt. $\angle$;
$\therefore$ $\angle$ s ABC, ABD are together = two rt. $\angle$ s.

Q. E. D.

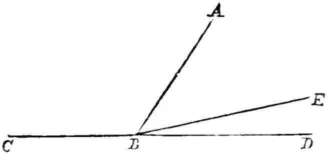
Ex. Straight lines drawn connecting the opposite angular points of a quadrilateral figure intersect each other in O. Shew that the angles at O are together equal to four right angles.

Note (1.) If two angles together make up a right angle, each is called the Complement of the other. Thus, in fig. 2, $\angle ABD$ is the complement of $\angle ABE$.

(2.) If two angles together make up two right angles, each is called the Supplement of the other. Thus, in both figures, $\angle ABD$ is the supplement of $\angle ABC$.

Proposition XIV. Theorem.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines must be in one and the same straight line.



At the pt. B in the st. line AB let the st. lines BC, BD, on opposite sides of AB, make $\angle$ s ABC, ABD together=two rt. angles.

Then BD must be in the same st. line with BC.

For if not, let BE be in the same st. line with BC.

Then $\angle$ s ABC, ABE together=two rt. $\angle$ s. I. 13.

And $\angle$ s ABC, ABD together=two rt. $\angle$ s. Hyp.

$\therefore$ sum of $\angle$ s ABC, ABE=sum of $\angle$ s ABC, ABD.

Take away from each of these equals the $\angle$ ABC;

then $\angle$ ABE= $\angle$ ABD, Ax. 3.

that is, the less=the greater; which is impossible,

$\therefore$ BE is not in the same st. line with BC.

Similarly it may be shewn that no other line but BD is in the same st. line with BC.

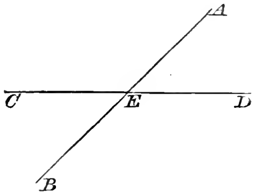
$\therefore$ BD is in the same st. line with BC.

Q. E. D.

Ex. Shew the necessity of the words *the opposite sides* in the enunciation.

Proposition XV. Theorem.

If two straight lines cut one another, the vertically opposite angles must be equal.



Let the st. lines AB, CD cut one another in the pt. E.

Then must $\angle\ AEC = \angle\ BED$ *and* $\angle\ AED = \angle\ BEC$.

For $\because AE$ meets CD,

$\therefore$ sum of $\angle$ s AEC, AED=two rt. $\angle$ s. I. 13.

And $\because DE$ meets AB,

$\therefore$ sum of $\angle$ s BED, AED=two rt. $\angle$ s; I. 13.

$\therefore$ sum of $\angle$ s AEC, AED=sum of $\angle$ s BED, AED;

$\therefore \angle\ AEC = \angle\ BED$. Ax. 3.

Similarly it may be shewn that $\angle\ AED = \angle\ BEC$.

Q. E. D.

Corollary I. From this it is manifest, that if two straight lines cut one another, the four angles, which they make at the point of intersection, are together equal to four right angles.

Corollary II. All the angles, made by any number of straight lines meeting in one point, are together equal to four right angles.

Ex. 1. Shew that the bisectors of AED and BEC are in the same straight line.

Ex. 2. Prove that $\angle\ AED$ is equal to the angle between two straight lines drawn at right angles from E to AE and EC, if both lie above CD.

Ex. 3. If AB, CD bisect each other in E; shew that the triangles AED, BEC are equal in all respects.

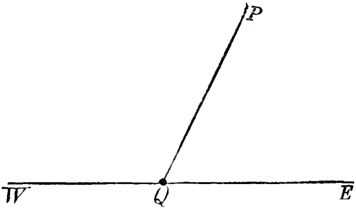
NOTE 3. *On Euclid's definition of an Angle.*

Euclid directs us to regard an angle as the inclination of two straight lines to each other, which meet, *but are not in the same straight line.*

Thus he does not recognise the existence of a single angle equal in magnitude to two right angles.

The words printed in italics are omitted as needless, in Def. VIII., p. 3, and that definition may be extended with advantage in the following terms :—

DEF. Let WQE be a fixed straight line, and QP a line which revolves about the fixed point Q, and which at first coincides with QE.



Then, when QP has reached the position represented in the diagram, we say that it has described the angle EQP.

When QP has revolved so far as to coincide with QW, we say that it has described an angle *equal to two right angles.*

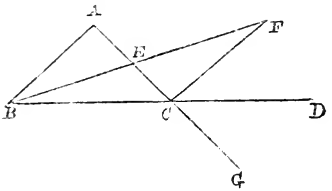
Hence we may obtain an easy proof of Prop. XIII. ; for whatever the position of PQ may be, the angles which it makes with WE are together equal to two right angles.

Again, in Prop. XV. it is evident that $\angle\, AED = \angle\, BEC$, since each has the same supplementary $\angle\, AEC$.

We shall shew hereafter, p. 149, how this definition may be extended, so as to embrace angles *greater than two right angles.*

Proposition XVI. Theorem.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.



Let the side BC of $\triangle ABC$ be produced to D.

Then must $\angle ACD$ *be greater than either* $\angle CAB$ *or* $\angle ABC$.

Bisect AC in E, and join BE. I. 10.

Produce BE to F, making $EF = BE$, and join FC.

Then in $\triangle$s BEA, FEC,

$\because BE = FE$, and $EA = EC$, and $\angle BEA = \angle FEC$, I. 15.

$\therefore \angle ECF = \angle EAB$. I. 4.

Now $\angle ACD$ is greater than $\angle ECF$; Ax. 9.

$\therefore \angle ACD$ is greater than $\angle EAB$,

that is, $\angle ACD$ is greater than $\angle CAB$.

Similarly, if AC be produced to G it may be shewn that

$\angle BCG$ is greater than $\angle ABC$.

and $\angle BCG = \angle ACD$; I. 15.

$\therefore \angle ACD$ is greater than $\angle ABC$.

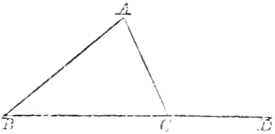
Q. E. D.

Ex. 1. From the same point there cannot be drawn more than two equal straight lines to meet a given straight line.

Ex. 2. If, from any point, a straight line be drawn to a given straight line making with it an acute and an obtuse angle, and if, from the same point, a perpendicular be drawn to the given line; the perpendicular will fall on the side of the acute angle.

Proposition XVII. Theorem.

Any two angles of a triangle are together less than two right angles.



Let ABC be any $\triangle$.

Then must any two of its $\angle$ s be together less than two rt. $\angle$ s.

Produce BC to D.

Then $\angle ACD$ is greater than $\angle ABC$. I. 16.

$\therefore$ $\angle$ s ACD, ACB are together greater than $\angle$ s ABC, ACB.

But $\angle$ s ACD, ACB together = two rt. $\angle$ s. I. 13

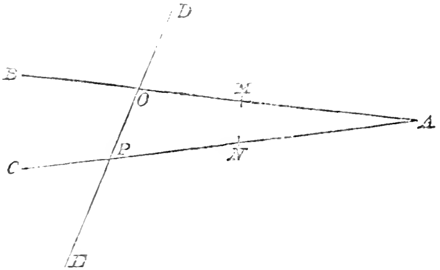
$\therefore$ $\angle$ s ABC, ACB are together less than two rt. $\angle$ s.

Similarly it may be shewn that $\angle$ s ABC, BAC and also that $\angle$ s BAC, ACB are together less than two rt. $\angle$ s.

Q. E. D.

Note 4. *On the Sixth Postulate.*

We learn from Prop. XVII. that if two straight lines BM and CN, which meet in A, are met by another straight line DE in the points O, P,



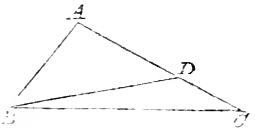
the angles MOP and NPO are together less than two right angles.

The Sixth Postulate asserts that if a line DE meeting two other lines BM, CN makes MOP, NPO, the two interior

angles on the same side of it, together less than two right angles, BM and CN should meet if produced on the same side of DE on which are the angles MOP and NPO.

Proposition XVIII. Theorem.

If one side of a triangle be greater than a second, the angle opposite the first must be greater than that opposite the second.



In $\triangle ABC$, let side AC be greater than AB.

Then must $\angle ABC$ be greater than $\angle ACB$.

From AC cut off $AD = AB$, and join BD. I. 3.

Then $\because AB = AD$,

$\therefore \angle ADB = \angle ABD$, I. A.

And $\because CD$, a side of $\triangle BDC$, is produced to A.

$\therefore \angle ADB$ is greater than $\angle ACB$; I. 16

$\therefore$ also $\angle ABD$ is greater than $\angle ACB$.

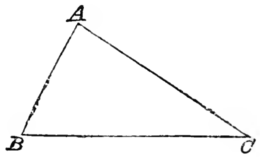
Much more is $\angle ABC$ greater than $\angle ACB$.

Q. E. D.

Ex. Shew that if two angles of a triangle be equal, the sides which subtend them are equal also (Eucl. I. 6).

Proposition XIX. Theorem.

If one angle of a triangle be greater than a second, the side opposite the first must be greater than that opposite the second.



In $\triangle ABC$, let $\angle ABC$ be greater than $\angle ACB$.

Then must AC be greater than AB.

For if AC be not greater than AB,

AC must either $= AB$, or be less than AB.

Now AC cannot $= AB$, for then I. A.

$\angle ABC$ would $= \angle ACB$, which is not the case.

And AC cannot be less than AB, for then I. 18.

$\angle ABC$ would be less than $\angle ACB$, which is not the case;

$\therefore$ AC is greater than AB.

Q. E. D.

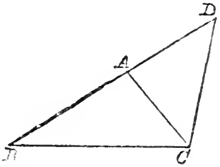
Ex. 1. In an obtuse-angled triangle, the greatest side is opposite the obtuse angle.

Ex. 2. BC, the base of an isosceles triangle BAC, is produced to any point D; shew that AD is greater than AB.

Ex. 3. The perpendicular is the shortest straight line, which can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than one more remote.

PROPOSITION XX. THEOREM.

Any two sides of a triangle are together greater than the third side.



Let ABC be a $\triangle$.

Then any two of its sides must be together greater than the third side.

Produce BA to D, making $AD = AC$, and join DC.

Then $\because AD = AC$,

$\therefore \angle ACD = \angle ADC$, that is, $\angle BDC$. I. A.

Now $\angle BCD$ is greater than $\angle ACD$;

$\therefore \angle BCD$ is also greater than $\angle BDC$;

$\therefore BD$ is greater than BC. I. 19.

But $BD = BA$ and AD together;

that is, $BD = BA$ and AC together;

$\therefore BA$ and AC together are greater than BC.

Similarly it may be shewn that

AB and BC together are greater than AC,

and BC and CA .... AB.

Q. E. D.

Ex. 1. Prove that any three sides of a quadrilateral figure are together greater than the fourth side.

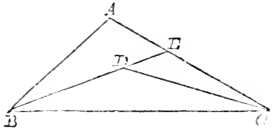
Ex. 2. Shew that any side of a triangle is greater than the difference between the other two sides.

Ex. 3. Prove that the sum of the distances of any point from the angular points of a quadrilateral is greater than half the perimeter of the quadrilateral.

Ex. 4. If one side of a triangle be bisected, the sum of the two other sides shall be more than double of the line joining the vertex and the point of bisection.

Proposition XXI. Theorem.

If, from the ends of the side of a triangle, there be drawn two straight lines to a point within the triangle; these will be together less than the other sides of the triangle, but will contain a greater angle.



Let ABC be a $\triangle$, and from D, a pt. in the $\triangle$, draw st. lines to B and C.

Then will BD, DC together be less than BA, AC, but $\angle BDC$ will be greater than $\angle BAC$.

Produce BD to meet AC in E.

Then BA, AE are together greater than BE. I. 20.

Add to each EC.

Then BA, AC are together greater than BE, EC.

Again, DE, EC are together greater than DC. I. 20.

Add to each BD.

Then BE, EC are together greater than BD, DC.

And it has been shewn that BA, AC are together greater than BE, EC;

$\therefore$ BA, AC are together greater than BD, DC.

Next, $\because$ $\angle BDC$ is greater than $\angle DEC$, I. 16.

and $\angle DEC$ is greater than $\angle BAC$, I. 16.

$\therefore$ $\angle BDC$ is greater than $\angle BAC$.

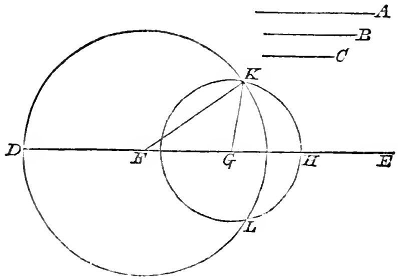
Q. E. D.

Ex. 1. Upon the base AB of a triangle ABC is described a quadrilateral figure $ADEB$, which is entirely within the triangle. Shew that the sides AC, CB of the triangle are together greater than the sides AD, DE, EB of the quadrilateral.

Ex. 2. Shew that the sum of the straight lines, joining the angles of a triangle with a point within the triangle, is less than the perimeter of the triangle, and greater than half the perimeter.

Proposition XXII. Problem.

To make a triangle, of which the sides shall be equal to three given straight lines, any two of which are together greater than the third.



Let A, B, C be the three given lines, any two of which are together greater than the third.

It is required to make a $\triangle$ having its sides $= A$, B, C respectively.

Take a st. line DE of unlimited length.

In DE make $DF = A$, $FG = B$, and $GH = C$. I. 3.

With centre F and distance FD, describe $\odot$ DKL.

With centre G and distance GH, describe $\odot$ HKL.

Join FK and GK.

Then $\triangle KFG$ has its sides $= A$, B, C respectively.

For $FK = FD$; Def. 13.

$\therefore FK = A$;

and $GK = GH$; Def. 13.

$\therefore GK = C$;

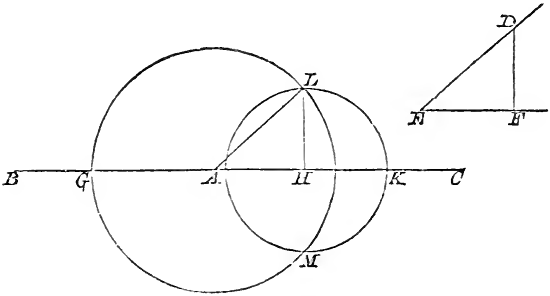
and $FG = B$;

$\therefore$ a $\triangle KFG$ has been described as reqd. Q. E. F.

Ex. Draw an isosceles triangle having each of the equal sides double of the base.

PROPOSITION XXIII. PROBLEM.

At a given point in a given straight line, to make an angle equal to a given angle.



Let A be the given pt., BC the given line, DEF the given $\angle$.

It is reqd. to make at pt. A an angle $= \angle DEF$.

In ED, EF take any pts. D. F; and join DF.

In AB, produced if necessary, make $AG=DE$.

In AC, produced if necessary, make $AH=EF$.

In HC, produced if necessary, make $HK=FD$.

With centre A, and distance AG, describe $\odot\, GLM$.

With centre H, and distance HK, describe $\odot\, LKM$.

Join AL and HL.

Then $\because LA=AG$, $\therefore LA=DE$; Ax. 1.

and $\because HL=HK$, $\therefore HL=FD$. Ax. 1.

Then in $\triangle$s LAH, DEF,

$\because LA=DE$, and $AH=EF$, and $HL=FD$;

$\therefore \angle LAH = \angle DEF$. I. c.

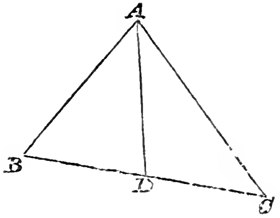
$\therefore$ an angle LAH has been made at pt. A as was reqd.

Q. E. F.

NOTE.—We here give the proof of a theorem, necessary to the proof of Prop. XXIV. and applicable to several propositions in Book III.

PROPOSITION D. THEOREM.

Every straight line, drawn from the vertex of a triangle to the base, is less than the greater of the two sides, or than either, if they be equal.



In the $\triangle ABC$, let the side AC be not less than AB.

Take any pt. D in BC, and join AD.

Then must AD be less than AC.

For $\because AC$ is not less than AB;

$\therefore \angle ABD$ is not less than $\angle ACD$. I. A. and 18.

But $\angle ADC$ is greater than $\angle ABD$; I. 16.

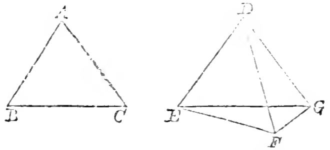
$\therefore \angle ADC$ is greater than $\angle ACD$;

$\therefore AC$ is greater than AD. I. 19.

Q. E. D.

PROPOSITION XXIV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; the base of that which has the greater angle must be greater than the base of the other.



In the $\triangle$s ABC, DEF,
let $AB = DE$ and $AC = DF$,
and let $\angle BAC$ be greater than $\angle EDF$.
Then must BC be greater than EF.

Of the two sides DE, DF let DE be not greater than DF.*
At pt. D in st. line ED make $\angle EDG = \angle BAC$, I. 23.
and make $DG = AC$ or DF, and join EG, GF.
Then $\because AB = DE$, and $AC = DG$, and $\angle BAC = \angle EDG$,
$\therefore BC = EG$, I. 4.
Again, $\because DG = DF$,
$\therefore \angle DFG = \angle DGF$; I. A.
$\therefore \angle EFG$ is greater than $\angle DGF$;
much more then $\angle EFG$ is greater than $\angle EGF$;
$\therefore EG$ is greater than EF. I. 19.
But $EG = BC$;
$\therefore BC$ is greater than EF.

Q. E. D.

* This line was added by Simson to obviate a defect in Euclid's proof. *Without* this condition, three distinct cases must be discussed. *With* the condition, we can prove that F must lie below EG.

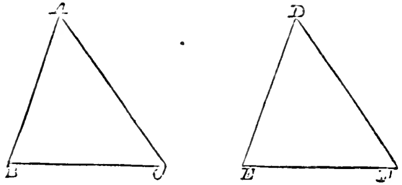
For since DF is not less than DE, and DG is drawn equal to DF, DG is not less than DE.

Hence by Prop. D, any line drawn from D to meet EG is less than DG, and therefore DF, being equal to DG, must extend beyond EG.

For another method of proving the Proposition, see p. 113.

Proposition XXV. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle also, contained by the sides of that which has the greater base, must be greater than the angle contained by the sides equal to them of the other.



In the $\triangle$s ABC, DEF,

let $AB=DE$ and $AC=DF$,

and let BC be greater than EF.

Then must $\angle BAC$ *be greater than* $\angle EDF$.

For $\angle BAC$ is greater than, equal to, or less than $\angle EDF$.

Now $\angle BAC$ cannot $= \angle EDF$,

for then, by I. 4, BC would $=EF$; which is not the case.

And $\angle BAC$ cannot be less than $\angle EDF$,

for then, by I. 24, BC would be less than EF; which is not the case;

$\therefore \angle BAC$ must be greater than $\angle EDF$.

Q. E. D.

Note.—In Prop. XXVI. Euclid includes two cases, in which two triangles are equal in all respects; viz., when the following parts are equal in the two triangles:

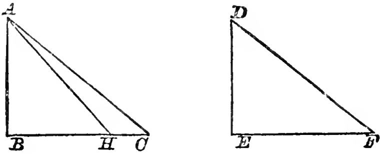
1. Two angles and the side between them.
2. Two angles and the side opposite one of them.

Of these we have already proved the first case, in Prop. B, so that we have only the second case left, to form the subject of Prop. XXVI., which we shall prove by the method of superposition.

For Euclid's proof of Prop. XXVI, see pp. 114-115.

Proposition XXVI. Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, those sides being opposite to equal angles in each; then must the triangles be equal in all respects.



In $\triangle$s ABC, DEF,

let $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$, and $AB = DE$.

Then must $BC = EF$, *and* $AC = DF$, *and* $\angle BAC = \angle EDF$.

Suppose $\triangle DEF$ to be applied to $\triangle ABC$,

so that D coincides with A, and DE falls on AB.

Then $\because DE = AB$, $\therefore E$ will coincide with B;

and $\because \angle DEF = \angle ABC$, $\therefore EF$ will fall on BC.

Then must F coincide with C: for, if not,

let F fall *between* B and C, at the pt. H. Join AH.

Then $\because \angle AHB = \angle DFE$, I. 4.

$\therefore \angle AHB = \angle ACB$,

the extr. $\angle$ = the intr. and opposite $\angle$, which is impossible.

$\therefore F$ does not fall between B and C.

Similarly, it may be shewn that F does not fall on BC produced.

$\therefore F$ coincides with C, and $\therefore BC = EF$;

$\therefore AC = DF$, and $\angle BAC = \angle EDF$, I. 4.

and $\therefore$ the triangles are equal in all respects.

Q. E. D.

Miscellaneous Exercises on Props. I. to XXVI.

1. M is the middle point of the base BC of an isosceles triangle ABC, and N is a point in AC. Shew that the difference between MB and MN is less than that between AB and AN.

2. ABC is a triangle, and the angle at A is bisected by a straight line which meets BC at D; shew that BA is greater than BD, and CA greater than CD.

3. AB, AC are straight lines meeting in A, and D is a given point. Draw through D a straight line cutting off equal parts from AB, AC.

4. Draw a straight line through a given point, to make equal angles with two given straight lines which meet.

5. A given angle BAC is bisected; if CA be produced to G and the angle BAG bisected, the two bisecting lines are at right angles.

6. Two straight lines are drawn to the base of a triangle from the vertex, one bisecting the vertical angle, and the other bisecting the base. Prove that the latter is the greater of the two lines.

7. Shew that Prop. XVII. may be proved without producing a side of the triangle.

8. Shew that Prop. XVIII. may be proved by means of the following construction: cut off $AD=AB$, draw AE, bisecting $\angle BAC$ and meeting BC in E, and join DE.

9. Shew that Prop. XX. can be proved, without producing one of the sides of the triangle, by bisecting one of the angles.

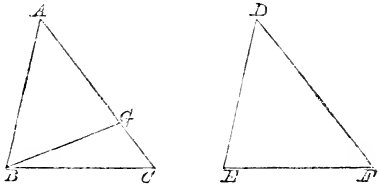
10. Given two angles of a triangle and the side adjacent to them, construct the triangle.

11. Shew that the perpendiculars, let fall on two sides of a triangle from any point in the straight line bisecting the angle contained by the two sides, are equal.

We conclude Section I. with the proof (omitted by Euclid) of another case in which two triangles are equal in all respects.

PROPOSITION E. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about a second angle in each equal: then, if the third angles in each be both acute, both obtuse, or if one of them be a right angle, the triangles are equal in all respects.



In the $\triangle$s ABC, DEF, let $\angle BAC = \angle EDF$, $AB=DE$, $BC=EF$, and let $\angle$s ACB, DFE be both acute, both obtuse, or let one of them be a right angle.

Then must $\triangle$s ABC, DEF be equal in all respects.

For if AC be not $=DF$, make $AG=DF$; and join BG.

Then in $\triangle$s BAG, EDF,

$\because BA=ED$, and $AG=DF$, and $\angle BAG = \angle EDF$,

$\therefore BG=EF$ and $\angle AGB = \angle DFE$. I. 4.

But $BC=EF$, and $\therefore BG=BC$;

$\therefore \angle BCG = \angle BGC$. I. A.

First, let $\angle ACB$ and $\angle DFE$ be both acute,

then $\angle AGB$ is acute, and $\therefore \angle BGC$ is obtuse; I. 13.

$\therefore \angle BCG$ is obtuse, which is contrary to the hypothesis.

Next, let $\angle ACB$ and $\angle DFE$ be both obtuse,

then $\angle AGB$ is obtuse, and $\therefore \angle BGC$ is acute; I. 13.

$\therefore \angle BCG$ is acute, which is contrary to the hypothesis.

Lastly, let one of the third angles ACB, DFE be a right angle.

If $\angle ACB$ be a rt. $\angle$,

then $\angle BGC$ is also a rt. $\angle$; I. A.

$\therefore \angle$s BCG, BGC together = two rt. $\angle$s, which is impossible. I. 17.

Again, if $\angle DFE$ be a rt. $\angle$,

then $\angle AGB$ is a rt. $\angle$, and $\therefore \angle BGC$ is a rt. $\angle$. I. 13.

Hence $\angle BCG$ is also a rt. $\angle$.

$\therefore \angle$s BCG, BGC together = two rt. $\angle$s, which is impossible, I. 17.

Hence AC is equal to DF,

and the $\triangle$s ABC, DEF are equal in all respects.

Q. E. D.

Cor. From the first case of this proposition we deduce the following important theorem:

If two right-angled triangles have the hypotenuse and one side of the one equal respectively to the hypotenuse and one side of the other, the triangles are equal in all respects.

Note. In the enunciation of Prop. E, if, instead of the words *if one of them be a right angle*, we put the words *both right angles*, this case of the proposition would be identical with I. 26.

SECTION II.

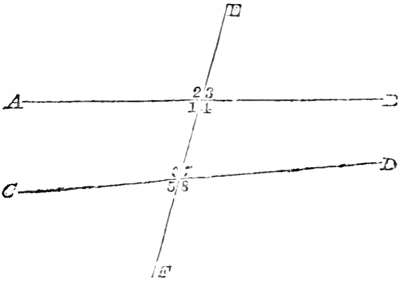
The Theory of Parallel Lines.

INTRODUCTION.

We have detached the Propositions, in which Euclid treats of Parallel Lines, from those which precede and follow them in the First Book, in order that the student may have a clearer notion of the difficulties attending this division of the subject, and of the way in which Euclid proposes to meet them.

We must first explain some technical terms used in this Section.

If a straight line EF cut two other straight lines AB, CD, it makes with these lines eight angles, to which particular names are given.



The angles numbered 1, 4, 6, 7 are called *Interior* angles.
.... 2, 3, 5, 8 .... *Exterior* ....

The angles marked 1 and 7 are called *alternate* angles.

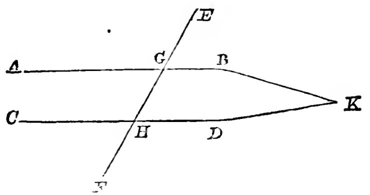
The angles marked 4 and 6 are also called alternate angles.

The pairs of angles 1 and 5, 2 and 6, 4 and 8, 3 and 7 are called *corresponding* angles.

Note. From I. 13 it is clear that the angles 1, 4, 6, 7 are together equal to four right angles.

Proposition XXVII. Theorem.

If a straight line, falling upon two other straight lines, make the alternate angles equal to one another; these two straight lines must be parallel.



Let the st. line EF, falling on the st. lines AB, CD, make the alternate $\angle$ s AGH, GHD equal.

Then must AB be $\parallel$ to CD.

For if not, AB and CD will meet, if produced, either towards B, D, or towards A, C.

Let them be produced and meet towards B, D in K.

Then GHK is a $\triangle$;

and $\therefore$ $\angle AGH$ is greater than $\angle GHD$. I. 16.

But $\angle AGH = \angle GHD$, Hyp.

which is impossible.

$\therefore$ AB, CD do not meet when produced towards B, D.

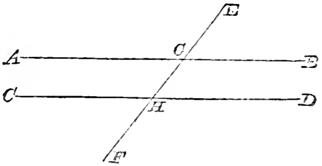
In like manner it may be shewn that they do not meet when produced towards A, C.

$\therefore$ AB and CD are parallel. Def. 26.

Q. E. D.

Proposition XXVIII. Theorem.

If a straight line, falling upon two other straight lines, make the exterior angle equal to the interior and opposite upon the same side of the line, or make the interior angles upon the same side together equal to two right angles; the two straight lines are parallel to one another.



Let the st. line EF, falling on st. lines AB, CD, make

I. $\angle\, EGB =$ corresponding $\angle\, GHD$, or

II. $\angle$ s BGH, GHD together = two rt. $\angle$ s.

Then, in either case, AB must be $\parallel$ to CD.

I. $\because \angle\, EGB$ is given $= \angle\, GHD$, Hyp.

and $\angle\, EGB$ is known to be $= \angle\, AGH$, I. 15.

$\therefore \angle\, AGH = \angle\, GHD$;

and these are alternate $\angle$ s;

$\therefore AB$ is $\parallel$ to CD. I. 27.

II. $\because \angle$ s BGH, GHD together = two rt. $\angle$ s, Hyp.

and $\angle$ s BGH, AGH together = two rt. $\angle$ s, I. 13.

$\therefore \angle$ s BGH, AGH together $= \angle$ s BGH, GHD together;

$\therefore \angle\, AGH = \angle\, GHD$;

$\therefore AB$ is $\parallel$ to CD. I. 27.

Q. E. D.

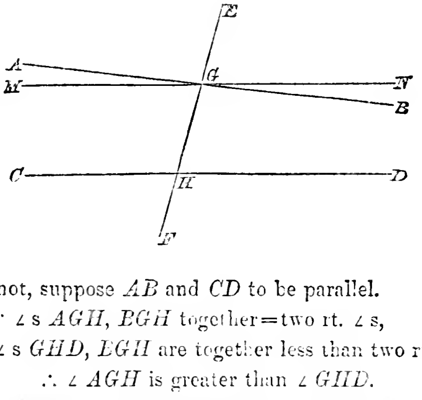
Note 5. *On the Sixth Postulate.*

In the place of Euclid's Sixth Postulate many modern writers on Geometry propose, as more evident to the senses, the following Postulate :—

"*Two straight lines which cut one another cannot* BOTH *be parallel to the same straight line.*"

If this be assumed, we can prove Post. 6, as a Theorem, thus :

Let the line EF falling on the lines AB, CD make the $\angle$ s BGH, GHD together less than two rt. $\angle$ s. Then must AB, CD meet when produced towards B, D.



For if not, suppose AB and CD to be parallel.

Then $\because$ $\angle$ s AGH, BGH together = two rt. $\angle$ s, I. 13.

and $\angle$ s GHD, BGH are together less than two rt. $\angle$ s,

$\therefore$ $\angle$ AGH is greater than $\angle$ GHD.

Make $\angle$ MGH = $\angle$ GHD, and produce MG to N.

Then $\because$ the alternate $\angle$ s MGH, GHD are equal,

$\therefore$ MN is $\parallel$ to CD. I. 27.

Thus two lines MN, AB which cut one another are both parallel to CD, which is impossible.

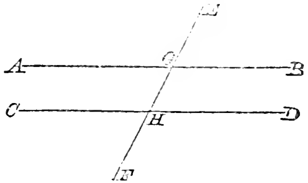
$\therefore$ AB and CD are not parallel.

It is also clear that they meet towards B, D, because GB lies between GN and HD.

Q. E. D

PROPOSITION XXIX. THEOREM.

If a straight line fall upon two parallel straight lines, it makes the two interior angles upon the same side together equal to two right angles, and also the alternate angles equal to one another, and also the exterior angle equal to the interior and opposite upon the same side.



Let the st. line EF fall on the parallel st. lines AB, CD.

Then must

I. $\angle$ s BGH, GHD together = two rt. $\angle$ s.

II. $\angle$ AGH = alternate $\angle$ GHD.

III. $\angle$ EGB = corresponding $\angle$ GHD.

I. $\angle$ s BGH, GHD cannot be together *less* than two rt. $\angle$ s,
for then AB and CD would meet if produced towards B and D, Post. 6.
which cannot be, for they are parallel.

Nor can $\angle$ s BGH, GHD be together *greater* than two rt. $\angle$ s,
for then $\angle$ s AGH, GHC would be together less than two rt. $\angle$ s, I. 13.
and AB, CD would meet if produced towards A and C Post. 6
which cannot be, for they are parallel,
$\therefore$ $\angle$ s BGH, GHD together = two rt. $\angle$ s.

II. $\because$ $\angle$ s BGH, GHD together = two rt. $\angle$ s,
and $\angle$ s BGH, AGH together = two rt. $\angle$ s, I. 13.
$\therefore$ $\angle$ s BGH, AGH together = $\angle$ s BGH, GHD together,
and $\therefore$ $\angle$ AGH = $\angle$ GHD. Ax. 3.

III. $\because$ $\angle$ AGH = $\angle$ GHD,
and $\angle$ AGH = $\angle$ EGB, I. 15.
$\therefore$ $\angle$ EGB = $\angle$ GHD. Ax. 1

Q. E. D.

EXERCISES.

1. If through a point, equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines; they will intercept equal portions of the parallel lines.

2. If a straight line be drawn, bisecting one of the angles of a triangle, to meet the opposite side; the straight lines drawn from the point of section, parallel to the other sides and terminated by those sides, will be equal.

3. If any straight line joining two parallel straight lines be bisected, any other straight line, drawn through the point of bisection to meet the two lines, will be bisected in that point.

NOTE. One Theorem (A) is said to be the *converse* of another Theorem (B), when the hypothesis in (A) is the conclusion in (B), and the conclusion in (A) is the hypothesis in (B).

For example, the Theorem I. A. may be stated thus :

Hypothesis. If two sides of a triangle be equal.

Conclusion. The angles opposite those sides must also be equal.

The converse of this is the Theorem I. B. Cor. :

Hypothesis. If two angles of a triangle be equal.

Conclusion. The sides opposite those angles must also be equal.

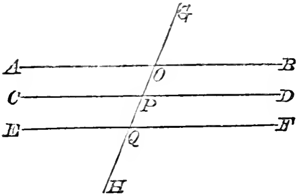
The following are other instances ;

Postulate VI. is the converse of I. 17.

I. 29 is the converse of I. 27 and 28.

Proposition XXX. Theorem.

Straight lines which are parallel to the same straight line are parallel to one another.



Let the st. lines AB, CD be each $\parallel$ to EF.

Then must AB be $\parallel$ to CD.

Draw the st. line GH, cutting AB, CD, EF in the pts. O, P, Q.

Then $\because$ GH cuts the $\parallel$ lines AB, EF,

$\therefore$ $\angle AOP$ = alternate $\angle PQF$. I. 29.

And $\because$ GH cuts the $\parallel$ lines CD, EF,

$\therefore$ extr. $\angle OPD$ = intr. $\angle PQF$; I. 29.

$\therefore$ $\angle AOP = \angle OPD$;

and these are alternate angles;

$\therefore$ AB is $\parallel$ to CD. I. 27.

Q. E. D.

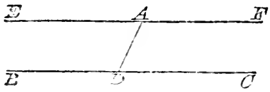
The following Theorems are important. They admit of easy proof, and are therefore left as Exercises for the student.

1. If two straight lines be parallel to two other straight lines, each to each, the first pair make the same angles with one another as the second.

2. If two straight lines be perpendicular to two other straight lines, each to each, the first pair make the same angles with one another as the second.

Proposition XXXI. Problem.

To draw a straight line through a given point parallel to a given straight line.



Let A be the given pt. and BC the given st. line.

It is required to draw through A a st. line $\parallel$ to BC.

In BC take any pt. D, and join AD.

Make $\angle DAE = \angle ADC$. I. 23.

Produce EA to F. Then EF shall be $\parallel$ to BC.

For $\because$ AD, meeting EF and BC, makes the alternate angles equal, that is, $\angle EAD = \angle ADC$,

$\therefore$ EF is $\parallel$ to BC. I. 27.

$\therefore$ a st. line has been drawn through A $\parallel$ to BC.

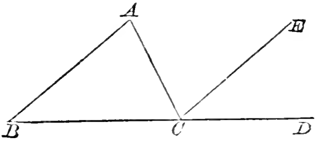
Q. E. F.

Ex. 1. From a given point draw a straight line, to make an angle with a given straight line that shall be equal to a given angle.

Ex. 2. Through a given point A draw a straight line ABC, meeting two parallel straight lines in B and C, so that BC may be equal to a given straight line.

Proposition XXXII. Theorem.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of every triangle are together equal to two right angles.



Let ABC be a $\triangle$, and let one of its sides, BC, be produced to D.

Then will

I. $\angle ACD = \angle$s ABC, BAC *together.*

II. $\angle$s ABC, BAC, ACB *together = two rt.* $\angle$*s.*

From C draw $CE \parallel$ to AB. I. 31.

Then I. $\because$ BD meets the $\parallel$s EC, AB,

$\therefore$ extr. $\angle ECD$ = intr. $\angle ABC$. I. 29.

And $\because$ AC meets the $\parallel$s EC, AB,

$\therefore$ $\angle ACE$ = alternate $\angle BAC$. I. 29.

$\therefore$ $\angle$s ECD, ACE together = $\angle$s ABC, BAC together;

$\therefore$ $\angle ACD = \angle$s ABC, BAC together.

And II. $\because$ $\angle$s ABC, BAC together = $\angle ACD$,

to each of these equals add $\angle ACB$;

then $\angle$s ABC, BAC, ACB together = $\angle$s ACD, ACB together,

$\therefore$ $\angle$s ABC, BAC, ACB together = two rt. $\angle$s. I. 13.

Q. E. D.

Ex. 1. In an acute-angled triangle, any two angles are greater than the third.

Ex. 2. The straight line, which bisects the external vertical angle of an isosceles triangle is parallel to the base.

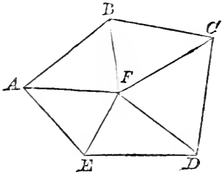
Ex. 3. If the side BC of the triangle ABC be produced to D, and AE be drawn bisecting the angle BAC and meeting BC in E; shew that the angles ABD, ACD are together double of the angle AED.

Ex. 4. If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet; shew that they will contain an angle equal to an exterior angle at the base of the triangle.

Ex. 5. If the straight line bisecting the external angle of a triangle be parallel to the base; prove that the triangle is isosceles.

The following Corollaries to Prop. 32 were first given in Simson's Edition of Euclid.

Cor. 1. *The sum of the interior angles of any rectilinear figure together with four right angles is equal to twice as many right angles as the figure has sides.*



Let $ABCDE$ be any rectilinear figure.

Take any pt. F within the figure, and from F draw the st. lines FA, FB, FC, FD, FE to the angular pts. of the figure

Then there are formed as many $\angle$s as the figure has sides.

The three $\angle$s in *each* of these $\triangle$s together = two rt. $\angle$s.

$\therefore$ *all* the $\angle$s in these $\triangle$s together = twice as many right $\angle$s as there are $\triangle$s, that is, twice as many right $\angle$s as the figure has sides.

Now angles of all the $\triangle$s = $\angle$s at A, B, C, D, E and $\angle$s at F,

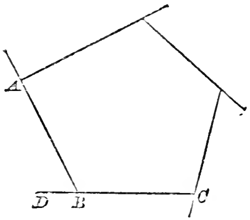
that is, = $\angle$s of the figure and $\angle$s at F,

and $\therefore$ = $\angle$s of the figure and four rt. $\angle$s. I. 15. Cor. 2

$\therefore$ $\angle$s of the figure and four rt. $\angle$s = twice as many rt. $\angle$s as the figure has sides.

Cor. 2. *The exterior angles of any convex rectilinear figure, made by producing each of its sides in succession, are together equal to four right angles.*

Every interior angle, as *ABC*, and its adjacent exterior angle, as *ABD*, together are = two rt. ∠ s.



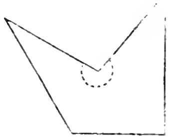
∴ all the intr. ∠ s together with all the extr. ∠ s
= twice as many rt. ∠ s as the figure has sides.

But all the intr. ∠ s together with four rt. ∠ s
= twice as many rt. ∠ s as the figure has sides.

∴ all the intr. ∠ s together with all the extr. ∠ s
= all the intr. ∠ s together with four rt. ∠ s.

∴ all the extr. ∠ s = four rt. ∠ s.

Note. The latter of these corollaries refers only to *convex* figures, that is, figures in which every interior angle is less than two right angles. When a figure contains an angle greater



than two right angles, as the angle marked by the dotted line in the diagram, this is called a reflex *angle*. See p. 149.

Ex. 1. The exterior angles of a quadrilateral made by producing the sides successively are together equal to the interior angles.

Ex. 2. Prove that the interior angles of a hexagon are equal to eight right angles.

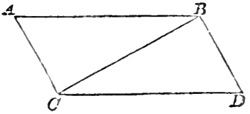
Ex. 3. Shew that the angle of an equiangular pentagon is $\frac{6}{5}$ of a right angle.

Ex. 4. How many sides has the rectilinear figure, the sum of whose interior angles is double that of its exterior angles?

Ex. 5. How many sides has an equiangular polygon, four of whose angles are together equal to seven right angles?

PROPOSITION XXXIII. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are also themselves equal and parallel.



Let the equal and ∥ st. lines AB, CD be joined towards the same parts by the st. lines AC, BD.

Then must AC and BD be equal and ∥.

Join BC.

Then $\because AB$ is ∥ to CD,

$\therefore \angle ABC =$ alternate $\angle DCB$. I. 29.

Then in $\triangle$s ABC, BCD,

$\because AB = CD$, and BC is common, and $\angle ABC = \angle DCB$,

$\therefore AC = BD$, and $\angle ACB = \angle DBC$. I. 4.

Then $\because BC$, meeting AC and BD,

makes the alternate $\angle$s ACB, DBC equal,

$\therefore AC$ is ∥ to BD.

Q. E. D.

Miscellaneous Exercises on Sections I. and II.

1. If two exterior angles of a triangle be bisected by straight lines which meet in O; prove that the perpendiculars from O on the sides, or the sides produced, of the triangle are equal.

2. Trisect a right angle.

3. The bisectors of the three angles of a triangle meet in one point.

4. The perpendiculars to the three sides of a triangle drawn from the middle points of the sides meet in one point.

5. The angle between the bisector of the angle BAC of the triangle ABC and the perpendicular from A on BC, is equal to half the difference between the angles at B and C.

6. If the straight line AD bisect the angle at A of the triangle ABC, and BDE be drawn perpendicular to AD, and meeting AC, or AC produced, in E; shew that BD is equal to DE.

7. Divide a right-angled triangle into two isosceles triangles.

8. AB, CD are two given straight lines. Through a point E between them draw a straight line GEH, such that the intercepted portion GH shall be bisected in E.

9. The vertical angle O of a triangle OPQ is a right, acute, or obtuse angle, according as OR, the line bisecting PQ, is equal to, greater or less than the half of PQ.

10. Shew by means of Ex. 9 how to draw a perpendicular to a given straight line from its extremity without producing it.

SECTION III.

On the Equality of Rectilinear Figures in respect of Area.

The amount of space enclosed by a Figure is called the Area of that figure.

Euclid calls two figures *equal* when they enclose the same amount of space. They may be dissimilar in shape, but if the areas contained within the boundaries of the figures be the same, then he calls the figures *equal*. He regards a triangle, for example, as a figure having sides and angles and area, and he proves in this section that two triangles may have equality of area, though the sides and angles of each may be unequal.

Coincidence of their boundaries is a test of the equality of all geometrical magnitudes, as we explained in Note 1, page 14.

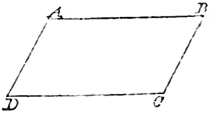
In the case of lines and angles it is the only test: in the case of *figures* it is *a test, but not the only test*; as we shall shew in this Section.

The sign =, standing between the symbols denoting two *figures*, must be read *is equal in area to.*

Before we proceed to prove the Propositions included in this Section, we must complete the list of Definitions required in Book I., continuing the numbers prefixed to the definitions in page 6.

Definitions.

XXVII. A Parallelogram is a four-sided figure whose opposite sides are parallel.

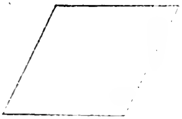


For brevity we often designate a parallelogram by two letters only, which mark opposite angles. Thus we call the figure in the margin the parallelogram *AC*.

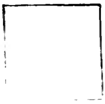
XXVIII. A Rectangle is a parallelogram, having one of its angles a right angle.

Hence by I. 29, *all* the angles of a rectangle are right angles.

XXIX. A Rhombus is a parallelogram, having its sides equal.



XXX. A Square is a parallelogram, having its sides equal and one of its angles a right angle.



Hence, by I. 29, *all* the angles of a square are right angles.

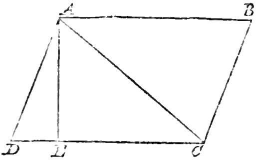
XXXI. A Trapezium is a four-sided figure of which two sides only are parallel.

XXXII. A Diagonal of a four-sided figure is the straight line joining two of the opposite angular points.

XXXIII. The Altitude of a Parallelogram is the perpendicular distance of one of its sides from the side opposite, regarded as the Base.

The altitude of a triangle is the perpendicular distance of one of its angular points from the side opposite, regarded as the base.

Thus if $ABCD$ be a parallelogram, and AE a perpendicular let fall from A to CD, AE is the *altitude* of the parallelogram, and also of the triangle ACD.



If a perpendicular be let fall from B to DC produced, meeting DC in F, BF is the altitude of the parallelogram.

Exercises.

Prove the following theorems:

1. The diagonals of a square make with each of the sides an angle equal to half a right angle.

2. If two straight lines bisect each other, the lines joining their extremities will form a parallelogram.

3. Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

4. If the straight lines joining two opposite angular points of a parallelogram bisect the angles, the parallelogram has all its sides equal.

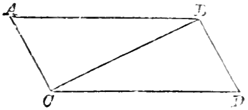
5. If the opposite angles of a quadrilateral be equal, the quadrilateral is a parallelogram.

6. If two opposite sides of a quadrilateral figure be equal to one another, and the two remaining sides be also equal to one another, the figure is a parallelogram.

7. If one angle of a rhombus be equal to two-thirds of two right angles, the diagonal drawn from that angular point divides the rhombus into two equilateral triangles.

Proposition XXXIV. Theorem.

The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects it.



Let $ABDC$ be a $\square$, and BC a diagonal of the $\square$.

Then must $AB=DC$ and $AC=DB$,

and $\angle BAC=\angle CDB$, and $\angle ABD=\angle ACD$

and $\triangle ABC=\triangle DCB$.

For $\because AB$ is $\parallel$ to CD, and BC meets them,

$\therefore \angle ABC=$alternate $\angle DCB$, I. 29

and $\because AC$ is $\parallel$ to BD, and BC meets them,

$\therefore \angle ACB=$alternate $\angle DBC$. I. 29.

Then in $\triangle$s ABC, DCB,

$\because \angle ABC=\angle DCB$, and $\angle ACB=\angle DBC$,

and BC is common, a side adjacent to the equal $\angle$s in each;

$\therefore AB=DC$, and $AC=DB$, and $\angle BAC=\angle CDB$,

and $\triangle ABC=\triangle DCB$. I. B.

Also $\because \angle ABC=\angle DCB$, and $\angle DBC=\angle ACB$,

$\therefore \angle$s ABC, DBC together$=\angle$s DCB, ACB together,

that is, $\angle ABD=\angle ACD$.

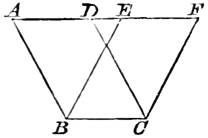
Q. E. D.

Ex. 1. Shew that the diagonals of a parallelogram bisect each other.

Ex. 2. Shew that the diagonals of a rectangle are equal.

Proposition XXXV. Theorem.

Parallelograms on the same base and between the same parallels are equal.



Let the ▱s *ABCD*, *EBCF* be on the same base *BC* and between the same ‖s *AF*, *BC*.

Then must ▱ *ABCD* = ▱ *EBCF*.

Case I. If *AD*, *EF* have no point common to both,

Then in the △s *FDC*, *EAB*,

∵ extr. ∠ *FDC* = intr. ∠ *EAB*, I. 29.

and intr. ∠ *DFC* = extr. ∠ *AEB*, I. 29.

and *DC* = *AB*, I. 34.

∴ △ *FDC* = △ *EAB*. I. 26.

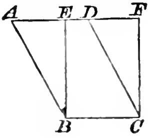
Now ▱ *ABCD* with △ *FDC* = figure *ABCF*;

and ▱ *EBCF* with △ *EAB* = figure *ABCF*;

∴ ▱ *ABCD* with △ *FDC* = ▱ *EBCF* with △ *EAB*;

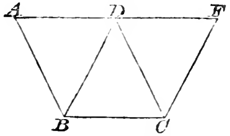
∴ ▱ *ABCD* = ▱ *EBCF*.

Case II. If the sides *AD*, *EF* overlap one another,



the same method of proof applies.

CASE III. If the sides opposite to BC be terminated in the same point D,



the same method of proof is applicable,
but it is easier to reason thus :

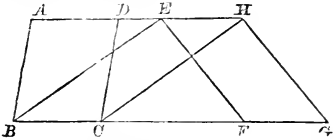
Each of the ▱s is double of $\triangle\ BDC$; I. 34.

$\therefore$ ▱ $ABCD$ = ▱ $DBCF$.

Q. E. D.

PROPOSITION XXXVI. THEOREM.

Parallelograms on equal bases, and between the same parallels, are equal to one another.



Let the ▱s $ABCD$, $EFGH$ be on equal bases BC, FG, and between the same ||s AH, BG.

Then must ▱ $ABCD$ = ▱ $EFGH$.

Join BE, CH.

Then $\because BC = FG$, Hyp.

and $EH = FG$; I. 34.

$\therefore BC = EH$;

and BC is || to EH. Hyp.

$\therefore EB$ is || to CH; I. 33.

$\therefore EBCH$ is a parallelogram.

Now ▱ $EBCH$ = ▱ $ABCD$, I. 35.

$\because$ they are on the same base BC and between the same ||s ;

and ▱ $EBCH$ = ▱ $EFGH$, I. 35.

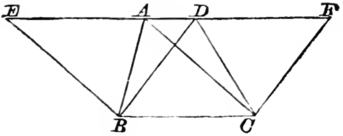
$\because$ they are on the same base EH and between the same ||s ,

$\therefore$ ▱ $ABCD$ = ▱ $EFGH$.

Q. E. D.

Proposition XXXVII. Theorem.

Triangles upon the same base, and between the same parallels, are equal to one another.



Let $\triangle$s ABC, DBC be on the same base BC and between the same $\parallel$s AD, BC.

Then must $\triangle\ ABC = \triangle\ DBC$.

From B draw $BE \parallel$ to CA to meet DA produced in E.

From C draw $CF \parallel$ to BD to meet AD produced in F.

Then $EBCA$ and $FCBD$ are parallelograms,

and $\square\ EBCA = \square\ FCBD$, I. 35.

$\because$ they are on the same base and between the same $\parallel$s.

Now $\triangle\ ABC$ is half of $\square\ EBCA$, I. 34.

and $\triangle\ DBC$ is half of $\square\ FCBD$; I. 34.

$\therefore\ \triangle\ ABC = \triangle\ DBC$. Ax. 7.

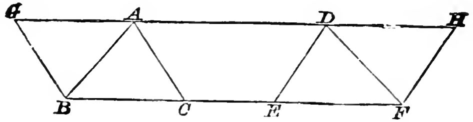
Q. E. D.

Ex. 1. If P be a point in a side AB of a parallelogram $ABCD$, and PC, PD be joined, the triangles PAD, PBC are together equal to the triangle PDC.

Ex. 2. If A, B be points in one, and C, D points in another of two parallel straight lines, and the lines AD, BC intersect in E, then the triangles AEC, BED are equal.

PROPOSITION XXXVIII. THEOREM.

Triangles upon equal bases, and between the same parallels, are equal to one another.



Let $\triangle$s *ABC, DEF* be on equal bases, *BC, EF*, and between the same ‖s *BF, AD*.

Then must $\triangle$ *ABC*= $\triangle$ *DEF*.

From *B* draw *BG* ‖ to *CA* to meet *DA* produced in *G*.

From *F* draw *FH* ‖ to *ED* to meet *AD* produced in *H*.

Then *CG* and *EH* are parallelograms, and they are equal,

∵ they are on equal bases *BC, EF*, and between the same ‖s *BF, GH*. I. 36

Now $\triangle$ *ABC* is half of ▱ *CG*,

and $\triangle$ *DEF* is half of ▱ *EH*;

∴ $\triangle$ *ABC*= $\triangle$ *DEF*. Ax. 7.

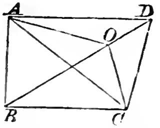
Q. E. D.

Ex. 1. Shew that a straight line, drawn from the vertex of a triangle to bisect the base, divides the triangle into two equal parts.

Ex. 2. In the equal sides *AB, AC* of an isosceles triangle *ABC* points *D, E* are taken such that *BD*=*AE*. Shew that the triangles *CBD, ABE* are equal.

Proposition XXXIX. Theorem.

Equal triangles upon the same base, and upon the same side of it, are between the same parallels.



Let the equal $\triangle$s ABC, DBC be on the same base BC, and on the same side of it.

Join AD.

Then must AD *be* $\parallel$ *to* BC.

For if not, through A draw $AO \parallel$ to BC, so as to meet BD, or BD produced, in O, and join OC.

Then $\because$ $\triangle$s ABC, OBC are on the same base and between the same $\parallel$s,

$\therefore \triangle ABC = \triangle OBC$. I. 37.

But $\triangle ABC = \triangle DBC$; Hyp.

$\therefore \triangle OBC = \triangle DBC$,

the less = the greater, which is impossible;

$\therefore AO$ is not $\parallel$ to BC.

In the same way it may be shewn that no other line passing through A but AD is $\parallel$ to BC;

$\therefore AD$ is $\parallel$ to BC.

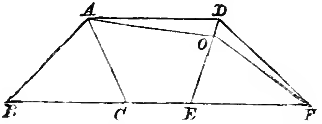
Q. E. D.

Ex. 1. AD is parallel to BC; AC, BD meet in E; BC is produced to P so that the triangle PEB is equal to the triangle ABC: shew that PD is parallel to AC.

Ex. 2. If of the four triangles into which the diagonals divide a quadrilateral, two opposite ones are equal, the quadrilateral has two opposite sides parallel.

Proposition XL. Theorem.

Equal triangles upon equal bases, in the same straight line, and towards the same parts, are between the same parallels.



Let the equal $\triangle$s ABC, DEF be on equal bases BC, EF in the same st. line BF and towards the same parts.

Join AD.

Then must AD be $\parallel$ to BF.

For if not, through A draw $AO \parallel$ to BF, so as to meet ED, or ED produced, in O, and join OF.

Then $\triangle ABC = \triangle OEF$, $\because$ they are on equal bases and between the same $\parallel$s. I. 38.

But $\triangle ABC = \triangle DEF$; Hyp.

$\therefore \triangle OEF = \triangle DEF$,

the less = the greater, which is impossible.

$\therefore AO$ is not $\parallel$ to BF.

In the same way it may be shewn that no other line passing through A but AD is $\parallel$ to BF,

$\therefore AD$ is $\parallel$ to BF.

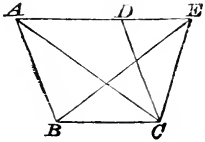
Q. E. D.

Ex. 1. The straight line, joining the points of bisection of two sides of a triangle, is parallel to the base, and is equal to half the base.

Ex. 2. The straight lines, joining the middle points of the sides of a triangle, divide it into four equal triangles.

Proposition XLI. Theorem.

If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram is double of the triangle.



Let the ▱ $ABCD$ and the △ EBC be on the same base BC and between the same ||s AE, BC.

Then must ▱ $ABCD$ be double of △ EBC.

Join AC.

Then △ ABC = △ EBC, ∵ they are on the same base and between the same ||s ; I. 37.

and ▱ $ABCD$ is double of △ ABC, ∵ AC is a diagonal of $ABCD$; I. 34.

∴ ▱ $ABCD$ is double of △ EBC.

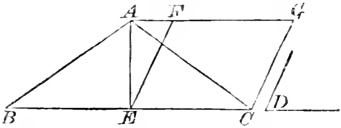
Q. E. D.

Ex. 1. If from a point, without a parallelogram, there be drawn two straight lines to the extremities of the two opposite sides, between which, when produced, the point does not lie, the difference of the triangles thus formed is equal to half the parallelogram.

Ex. 2. The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of its opposite sides, are together half of the parallelogram.

Proposition XLII. Problem.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.



Let ABC be the given $\triangle$, and D the given $\angle$.

It is required to describe a $\square$ equal to $\triangle$ ABC, having one of its $\angle$ s $= \angle$ D.

Bisect BC in E and join AE. I. 10.

At E make $\angle$ $CEF = \angle$ D. I. 23.

Draw AFG $\parallel$ to BC, and from C draw CG $\parallel$ to EF.

Then $FECG$ is a parallelogram.

Now $\triangle$ $AEB = \triangle$ AEC,

$\because$ they are on equal bases and between the same $\parallel$s. I. 38.

$\therefore$ $\triangle$ ABC is double of $\triangle$ AEC.

But $\square$ $FECG$ is double of $\triangle$ AEC,

$\because$ they are on same base and between same $\parallel$s. I. 41.

$\therefore$ $\square$ $FECG = \triangle$ ABC; Ax. 6.

and $\square$ $FECG$ has one of its $\angle$ s, $CEF = \angle$ D.

$\therefore$ $\square$ $FECG$ has been described as was reqd.

Q. E. F.

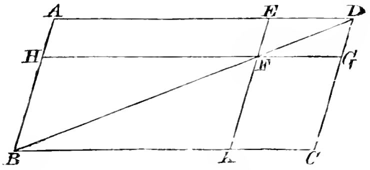
Ex. 1. Describe a triangle, which shall be equal to a given parallelogram, and have one of its angles equal to a given rectilineal angle.

Ex. 2. Construct a parallelogram, equal to a given triangle, and such that the sum of its sides shall be equal to the sum of the sides of the triangle.

Ex. 3. The perimeter of an isosceles triangle is greater than the perimeter of a rectangle, which is of the same altitude with, and equal to, the given triangle.

Proposition XLIII. Theorem.

The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.



Let $ABCD$ be a $\square$, of which BD is a diagonal, and EG, HK the $\square$s about BD, that is, through which BD passes,

and AF, FC the other $\square$s, which make up the whole figure $ABCD$,

and which are $\therefore$ called the Complements.

Then must complement AF=complement FC.

For $\because$ BD is a diagonal of $\square$ AC,

$\therefore$ $\triangle\ ABD = \triangle\ CDB$; I. 34.

and $\because$ BF is a diagonal of $\square$ HK,

$\therefore$ $\triangle\ HBF = \triangle\ KFB$; I. 34.

and $\because$ FD is a diagonal of $\square$ EG,

$\therefore$ $\triangle\ EFD = \triangle\ GDF$. I. 34.

Hence sum of $\triangle$s HBF, EFD=sum of $\triangle$s KFB, GDF

Take these equals from $\triangle$s ABD, CDB respectively,

then remaining $\square$ AF=remaining $\square$ FC. Ax. 3.

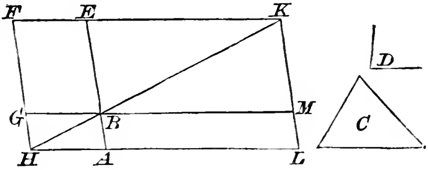
Q. E. D.

Ex. 1. If through a point O, within a parallelogram $ABCD$, two straight lines are drawn parallel to the sides, and the parallelograms OB, OD are equal; the point O is in the diagonal AC.

Ex. 2. $ABCD$ is a parallelogram, AMN a straight line meeting the sides BC, CD (one of them being produced) in M, N. Shew that the triangle MBN is equal to the triangle MDC.

Proposition XLIV. Problem.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given angle.



Let AB be the given st. line, C the given $\triangle$, D the given $\angle$.

It is required to apply to AB a $\square = \triangle C$ and having one of its $\angle$s $= \angle D$.

Make a $\square = \triangle C$, and having one of its angles $= \angle D$, I. 42.

and suppose it to be removed to such a position that one of the sides containing this angle is in the same st. line with AB, and let the $\square$ be denoted by $BEFG$.

Produce FG to H, draw $AH \parallel$ to BG or EF, and join BH.

Then $\because FH$ meets the $\parallel$s AH, EF,

$\therefore$ sum of $\angle$s AHF, HFE=two rt. $\angle$s ; I. 29.

$\therefore$ sum of $\angle$s BHG, HFE is less than two rt. $\angle$s ;

$\therefore$ HB, FE will meet if produced towards B, E. Post. 6.

Let them meet in K.

Through K draw $KL \parallel$ to EA or FH,

and produce HA, GB to meet KL in the pts. L, M.

Then $HFKL$ is a $\square$, and HK is its diagonal ;

and AG, ME are $\square$s about HK,

$\therefore$ complement BL=complement BF, I. 43.

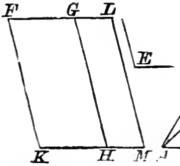
$\therefore \square BL = \triangle C$.

Also the $\square$ BL has one of its $\angle$s, $ABM = \angle EBG$, and $\therefore$ equal to $\angle D$.

Q. E. F.

PROPOSITION XLV. PROBLEM.

To describe a parallelogram, which shall be equal to a given rectilinear figure, and have one of its angles equal to a given angle.



Let $ABCD$ be the given rectil. figure, and E the given $\angle$.

It is required to describe a $\square$ = *to* $ABCD$, *having one of its* $\angle$ *s* = $\angle$ E.

Join AC.

Describe a $\square$ $FGHK = \triangle ABC$, having $\angle FKH = \angle E$. I. 42.

To GH apply a $\square$ $GHML = \triangle CDA$, having $\angle GHM = \angle E$. I. 44.

Then $FKML$ is the $\square$ reqd.

For $\because$ $\angle GHM$ and $\angle FKH$ are each $= \angle E$;

$\therefore$ $\angle GHM = \angle FKH$,

$\therefore$ sum of $\angle$ s GHM, GHK = sum of $\angle$ s FKH, GHK = two rt. $\angle$ s; I. 29.

$\therefore$ KHM is a st. line. I. 14.

Again, $\because$ HG meets the ||s FG, KM,

$\angle FGH = \angle GHM$,

$\therefore$ sum of $\angle$ s FGH, LGH = sum of $\angle$ s GHM, LGH = two rt. $\angle$ s; I. 29.

$\therefore$ FGL is a st. line. I. 14.

Then $\because$ KF is || to HG, and HG is || to LM

$\therefore$ KF is || to LM; I. 30.

and KM has been shewn to be || to FL,

$\therefore$ $FKML$ is a parallelogram,

and $\because$ $FH = \triangle ABC$, and $GM = \triangle CDA$,

$\therefore$ $\square$ FM = whole rectil. fig. $ABCD$,

and $\square$ FM has one of its $\angle$ s, $FKM = \angle E$.

In the same way a $\square$ may be constructed equal to a given rectil. fig. of any number of sides, and having one of its angles equal to a given angle. Q. E. F.

Miscellaneous Exercises.

1. If one diagonal of a quadrilateral bisect the other, it divides the quadrilateral into two equal triangles.

2. If from any point in the diagonal, or the diagonal produced, of a parallelogram, straight lines be drawn to the opposite angles, they will cut off equal triangles.

3. In a trapezium the straight line, joining the middle points of the parallel sides, bisects the trapezium.

4. The diagonals AC, BD of a parallelogram intersect in O, and P is a point within the triangle AOB; prove that the difference of the triangles CPD, APD is equal to the sum of the triangles APC, BPD.

5. If either diagonal of a parallelogram be equal to a side of the figure, the other diagonal shall be greater than any side of the figure.

6. If through the angles of a parallelogram four straight lines be drawn parallel to its diagonals, another parallelogram will be formed, the area of which will be double that of the original parallelogram.

7. If two triangles have two sides respectively equal and the included angles supplemental, the triangles are equal.

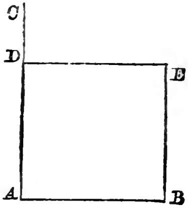
8. Bisect a given triangle by a straight line drawn from a given point in one of the sides.

9. The base AB of a triangle ABC is produced to a point D such that BD is equal to AB, and straight lines are drawn from A and D to E, the middle point of BC; prove that the triangle ADE is equal to the triangle ABC.

10. Prove that a pair of the diagonals of the parallelograms, which are about the diameter of any parallelogram, are parallel to each other.

Proposition XLVI. Problem.

To describe a square upon a given straight line.



Let AB be the given st. line.

It is required to describe a square on AB.

From A draw $AC \perp$ to AB. I. 11. Cor.

In AC make $AD = AB$.

Through D draw $DE \parallel$ to AB. I. 31.

Through B draw $BE \parallel$ to AD. I. 31.

Then AE is a parallelogram,

and $\therefore AB = ED$, and $AD = BE$. I. 34.

But $AB = AD$;

$\therefore AB, BE, ED, DA$ are all equal;

$\therefore AE$ is equilateral.

And $\angle BAD$ is a right angle.

$\therefore AE$ is a square, Def. XXX.

and it is described on AB.

Q. E. F.

Ex. 1. Shew how to construct a rectangle whose sides are equal to two given straight lines.

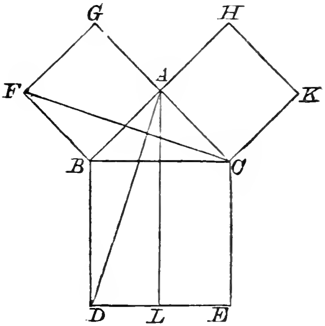
Ex. 2. Shew that the squares on equal straight lines are equal.

Ex. 3. Shew that equal squares must be on equal straight lines.

Note. The theorems in Ex. 2 and 3 are assumed by Euclid in the proof of Prop. XLVIII.

Proposition XLVII. Theorem.

In any right-angled triangle the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.



Let ABC be a right-angled $\triangle$, having the rt. $\angle\ BAC$.

Then must sq. on $BC=$ sum of sqq. on BA, AC.

On BC, CA, AB descr. the sqq. $BDEC$, $CKHA$, $AGFB$.

Through A draw $AL \parallel$ to BD or CE, and join AD, FC.

Then $\because$ $\angle\ BAC$ and $\angle\ BAG$ are both rt. $\angle$ s,

$\therefore$ CAG is a st. line ; I. 14

and $\because$ $\angle\ BAC$ and $\angle\ CAH$ are both rt. $\angle$ s ;

$\therefore$ BAH is a st. line. I. 14

Now $\because$ $\angle\ DBC = \angle\ FBA$, each being a rt. $\angle$,

adding to each $\angle\ ABC$, we have

$\angle\ ABD = \angle\ FBC.$ Ax. 2.

Then in $\triangle$ s ABD, FBC,

$\because$ $AB=FB$, and $BD=BC$, and $\angle\ ABD = \angle\ FBC$,

$\therefore$ $\triangle\ ABD = \triangle\ FBC.$ I. 4.

Now $\square$ BL is double of $\triangle\ ABD$, on same base BD and between same $\parallel$s AL, BD, I. 41.

and sq. BG is double of $\triangle\ FBC$, on same base FB and between same $\parallel$s FB, GC ; I. 41

$\therefore$ $\square$ $BL=$sq. BG.

Similarly, by joining AE, BK it may be shewn that

$$\square\ CL = \text{sq. } AK.$$

Now sq. on BC = sum of $\square\ BL$ and $\square\ CL$,

= sum of sq. BG and sq. AK,

= sum of sqq. on BA and AC.

Q. E. D.

Ex. 1. Prove that the square, described upon the diagonal of any given square, is equal to twice the given square.

Ex. 2. Find a line, the square on which shall be equal to the sum of the squares on three given straight lines.

Ex. 3. If one angle of a triangle be equal to the sum of the other two, and one of the sides containing this angle being divided into four equal parts, the other contains three of those parts; the remaining side of the triangle contains five such parts.

Ex. 4. The triangles ABC, DEF, having the angles ACB, DFE right angles, have also the sides AB, AC equal to DE, DF, each to each; shew that the triangles are equal in every respect.

Note. This Theorem has been already deduced as a Corollary from Prop. E, page 43.

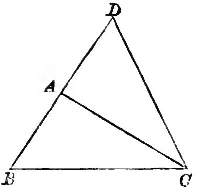
Ex. 5. Divide a given straight line into two parts, so that the square on one part shall be double of the square on the other.

Ex. 6. If from one of the acute angles of a right-angled triangle a line be drawn to the opposite side, the squares on that side and on the line so drawn are together equal to the sum of the squares on the segment adjacent to the right angle and on the hypotenuse.

Ex. 7. In any triangle, if a line be drawn from the vertex at right angles to the base, the difference between the squares on the sides is equal to the difference between the squares on the segments of the base.

PROPOSITION XLVIII. THEOREM.

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by those sides is a right angle.



Let the sq. on BC, a side of $\triangle ABC$, be equal to the sum of the sqq. on AB, AC.

Then must $\angle BAC$ *be a rt. angle.*

From pt. A draw $AD \perp$ to AC. I. 11.

Make $AD = AB$, and join DC.

Then $\because AD = AB$,

$\therefore$ sq. on AD = sq. on AB; I. 46, Ex. 2.

add to each sq. on AC.

then sum of sqq. on AD, AC = sum of sqq. on AB, AC.

But $\because \angle DAC$ is a rt. angle,

$\therefore$ sq. on DC = sum of sqq. on AD, AC; I. 47.

and, by hypothesis,

sq. on BC = sum of sqq. on AB, AC;

$\therefore$ sq. on DC = sq. on BC;

$\therefore DC = BC$. I. 46, Ex. 3.

Then in $\triangle$s ABC, ADC,

$\because AB = AD$, and AC is common, and $BC = DC$,

$\therefore \angle BAC = \angle DAC$; I. c.

and $\angle DAC$ is a rt. angle, by construction;

$\therefore \angle BAC$ is a rt. angle.

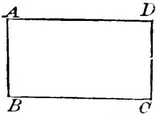
Q. E. D.

BOOK II.

INTRODUCTORY REMARKS.

THE geometrical figure with which we are chiefly concerned in this book is the RECTANGLE. A rectangle is said to be *contained by* any two of its adjacent sides.

Thus if $ABCD$ be a rectangle, it is said to be contained by AB, AD, or by any other pair of adjacent sides.



We shall use the abbreviation *rect.* AB, AD to express the words "the rectangle contained by AB, AD."

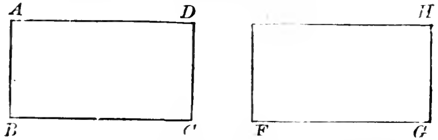
We shall make frequent use of a Theorem (employed, but not demonstrated, by Euclid) which may be thus stated and proved.

PROPOSITION A. THEOREM.

If the adjacent sides of one rectangle be equal to the adjacent sides of another rectangle, each to each, the rectangles are equal in area.

Let $ABCD$, $EFGH$ be two rectangles:

and let $AB=EF$ and $BC=FG$.



Then must rect. $ABCD=$ *rect.* $EFGH$.

For if the rect. $EFGH$ be applied to the rect. $ABCD$, so that EF coincides with AB,

then FG will fall on BC, $\because \angle EFG = \angle ABC$,

and G will coincide with C, $\because BC=FG$.

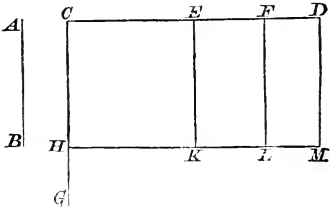
Similarly it may be shewn that H will coincide with D,

$\therefore$ rect. $EFGH$ coincides with and is therefore equal to rect $ABCD$.

Q. E. D.

PROPOSITION I. THEOREM.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several parts of the divided line.



Let AB and CD be two given st. lines,

and let CD be divided into any parts in E, F.

Then must rect. AB, CD = *sum of rect.* AB, CE *and rect.* AB, EF *and rect.* AB, FD.

From C draw $CG \perp$ to CD, and in CG make $CH = AB$.

Through H draw $HM \parallel$ to CD. I. 31.

Through E, F, and D draw EK, FL, $DM \parallel$ to CH.

Then EK and FL, being each $= CH$, are each $= AB$.

Now CM = sum of CK and EL and FM.

And CM = rect. AB, CD, $\because CH = AB$,

CK = rect. AB, CE, $\because CH = AB$,

EL = rect. AB, EF, $\because EK = AB$,

FM = rect. AB, FD, $\because FL = AB$;

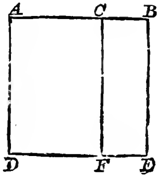
$\therefore$ rect. AB, CD = sum of rect. AB, CE and rect. AB, EF and rect. AB, FD.

Q. E. D.

Ex. If two straight lines be each divided into any number of parts, the rectangle contained by the two lines is equal to the rectangles contained by all the parts of the one taken separately with all the parts of the other.

Proposition II. Theorem.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.



Let the st. line AB be divided into any two parts in C.

Then must

sq. on AB = sum of rect. AB, AC and rect. AB, CB.

On AB describe the sq. $ADEB$ I. 46.

Through C draw $CF \parallel$ to AD. I. 31.

Then AE = sum of AF and CE.

Now AE is the sq. on AB,

AF = rect. AB, AC, $\because AD = AB$,

CE = rect. AB, CB, $\because BE = AB$,

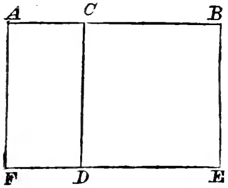
$\therefore$ sq. on AB = sum of rect. AB, AC and rect. AB, CB.

Q. E. D.

Ex. The square on a straight line is equal to four times the square on half the line.

Proposition III. Theorem.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts together with the square on the aforesaid part.



Let the st. line AB be divided into any two parts in C.

Then must

rect. AB, CB = sum of rect. AC, CB and sq. on CB.

On CB describe the sq. $CDEB$. I. 46

From A draw $AF \parallel$ to CD, meeting ED produced in F.

Then AE = sum of AD and CE.

Now AE = rect. AB, CB, $\because BE = CB$,

AD = rect. AC, CB, $\because CD = CB$,

CE = sq. on CB.

$\therefore$ rect. AB, CB = sum of rect. AC, CB and sq. on CB.

Q. E. D.

Note. When a straight line is cut in a point, the distances of the point of section from the ends of the line are called the *segments* of the line.

If a line AB be divided in C,

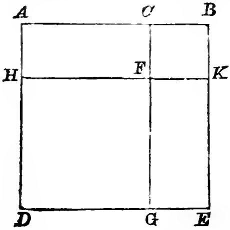
AC and CB are called the *internal* segments of AB.

If a line AC be produced to B,

AB and CB are called the *external* segments of AC.

Proposition IV. Theorem.

If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts together with twice the rectangle contained by the parts.



Let the st. line AB be divided into any two parts in C.

Then must

sq. on AB = sum of sqq. on AC, CB and twice rect. AC, CB.

On AB describe the sq. $ADEB$. I. 46.

From AD cut off $AH=CB$. Then $HD=AC$.

Draw $CG \parallel$ to AD, and $HK \parallel$ to AB, meeting CG in F.

Then $\because BK=AH$, $\therefore BK=CB$, Ax. I.

$\therefore BK, KF, FC, CB$ are all equal; and KBC is a rt. $\angle$;

$\therefore CK$ is the sq. on CB. Def. XXX.

Also HG=sq. on AC, $\because HF$ and HD each $=AC$.

Now AE=sum of HG, CK, AF, FE,

and AE=sq. on AB,

HG=sq. on AC,

CK=sq. on CB,

AF=rect. AC, CB, $\because CF=CB$,

FE=rect. AC, CB, $\because FG=AC$ and $FK=CB$.

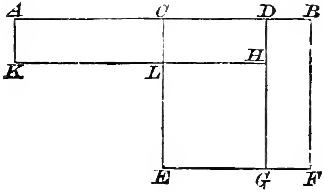
$\therefore$ sq. on AB=sum of sqq. on AC, CB and twice rect. AC, CB.

Q. E. D.

Ex. In a triangle, whose vertical angle is a right angle, a straight line is drawn from the vertex perpendicular to the base. Shew that the rectangle, contained by the segments of the base, is equal to the square on the perpendicular.

PROPOSITION V. THEOREM.

If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.



Let the st. line AB be divided equally in C and unequally in D.

Then must

rect. AD, DB together with sq. on CD = sq. on CB.

On CB describe the sq. $CEFB$. I. 46.

Draw $DG \parallel$ to CE, and from it cut off $DH = DB$. I. 31.

Draw $HLK \parallel$ to AD, and $AK \parallel$ to DH. I. 31.

Then rect. DF = rect. AL, $\because BF = AC$, and $BD = CL$.

Also LG = sq. on CD, $\because LH = CD$, and $HG = CD$.

Then rect. AD, DB together with sq. on CD

$= AH$ together with LG

= sum of AL and CH and LG

= sum of DF and CH and LG

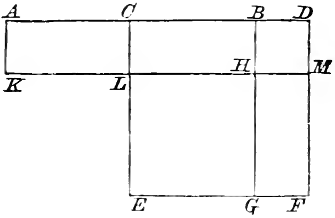
$= CF$

= sq. on CB.

Q. E. D.

Proposition VI. Theorem.

If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.



Let the st. line AB be bisected in C and produced to D.

Then must

rect. AD, DB together with sq. on CB=sq. on CD.

On CD describe the sq. $CEFD$. I. 46.

Draw $BG \parallel$ to CE, and cut off $BH=BD$. I. 31

Through H draw $KLM \parallel$ to AD I. 31.

Through A draw $AK \parallel$ to CE.

Now $\because BG=CD$ and $BH=BD$;

$\therefore HG=CB$; Ax. 3.

$\therefore$ rect. MG=rect. AL. II. A.

Then rect. AD, DB together with sq. on CB

=sum of AM and LG

=sum of AL and CM and LG

=sum of MG and CM and LG

$=CF$

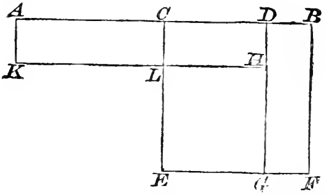
=sq. on CD.

Q. E. D.

NOTE. We here give the proof of an important theorem, which is usually placed as a corollary to Proposition V.

PROPOSITION B. THEOREM.

The difference between the squares on any two straight lines is equal to the rectangle contained by the sum and difference of those lines.



Let AC, CD be two st. lines, of which AC is the greater, and let them be placed so as to form one st. line AD.

Produce AD to B, making $CB=AC$.

Then $AD=$the sum of the lines AC, CD,

and $DB=$the difference of the lines AC, CD.

Then must difference between sqq. on AC, $CD=$rect. AD, DB.

On CB describe the sq. $CEFB$. I. 46.

Draw $DG \parallel$ to CE, and from it cut off $DH=DB$. I. 31.

Draw $HLK \parallel$ to AD, and $AK \parallel$ to DH. I. 31.

Then rect. $DF=$rect. AL, $\because BF=AC$, and $BD=CL$.

Also $LG=$sq. on CD, $\because LH=CD$, and $HG=CD$.

Then difference between sqq. on AC, CD

$=$difference between sqq. on CB, CD

$=$sum of CH and DF

$=$sum of CH and AL

$=AH$

$=$rect. AD, DH

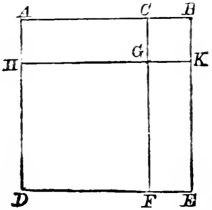
$=$rect. AD, DB.

Q. E. D.

Ex. Shew that Propositions V. and VI. might be deduced from this Proposition.

PROPOSITION VII. THEOREM.

If a straight line be divided into any two parts, the squares on the whole line and on one of the parts are equal to twice the rectangle contained by the whole and that part together with the square on the other part.



Let AB be divided into any two parts in C.

Then must

sqq. on AB, BC = twice rect. AB, BC together with sq. on AC.

On AB describe the sq. $ADEB$. I. 46.

From AD cut off $AH = CB$.

Draw $CF \parallel$ to AD and $HGK \parallel$ to AB. I. 31.

Then HF = sq. on AC, and CK = sq. on CB.

Then sqq. on AB, BC = sum of AE and CK

= sum of AK, HF, GE and CK

= sum of AK, HF and CE.

Now AK = rect. AB, BC, $\because BK = BC$;

CE = rect. AB, BC, $\because BE = AB$;

HF = sq. on AC.

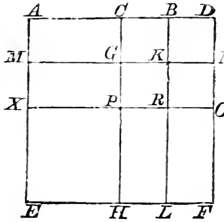
$\therefore$ sqq. on AB, BC = twice rect. AB, BC together with sq. on AC.

Q. E. D.

Ex. If straight lines be drawn from G to B and from G to D, shew that BGD is a straight line.

Proposition VIII. Theorem.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and the first part.



Let the st. line AB be divided into any two parts in C.

Produce AB to D, so that $BD=BC$.

Then must four times rect. AB, BC together with sq. on AC=sq. on AD.

On AD describe the sq. $AEFD$. I. 46.

From AE cut off AM and MX each$=CB$.

Through C, B draw CH, BL $\parallel$ to AE. I. 31.

Through M, X draw $MGKN$, $XPRO$ $\parallel$ to AD. I. 31.

Now $\because$ $XE=AC$, and $XP=AC$, $\therefore$ XH=sq. on AC.

Also $AG=MP=PL=RF$, II. A.

and $CK=GR=BN=KO$; II. A.

$\therefore$ sum of these eight rectangles

=four times the sum of AG, CK

=four times AK

=four times rect. AB, BC.

Then four times rect. AB, BC and sq. on AC

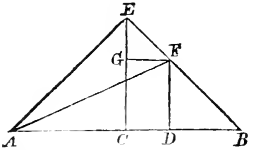
=sum of the eight rectangles and XH

$=AEFD$

=sq. on AD. Q. E. D.

Proposition IX. Theorem.

If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.



Let AB be divided equally in C and unequally in D.

Then must

sum of sqq. on AD, DB = *twice sum of sqq. on* AC, CD.

Draw $CE = AC$ at rt. $\angle$ s to AB, and join EA, EB.

Draw DF at rt. $\angle$ s to AB, meeting EB in F.

Draw FG at rt. $\angle$ s to EC, and join AF

Then $\because \angle ACE$ is a rt. $\angle$,

$\therefore$ sum of $\angle$ s AEC, EAC = a rt. $\angle$; I. 32.

and $\because \angle AEC = \angle EAC$, I. A.

$\therefore \angle AEC$ = half a rt. $\angle$.

So also $\angle BEC$ and $\angle EBC$ are each = half a rt. $\angle$.

Hence $\angle AEF$ is a rt. $\angle$.

Also, $\because \angle GEF$ is half a rt. $\angle$, and $\angle EGF$ is a rt. $\angle$;

$\therefore \angle EFG$ is half a rt. $\angle$;

$\therefore \angle EFG = \angle GEF$, and $\therefore EG = GF$. I. B. Cor.

So also $\angle BFD$ is half a rt. $\angle$, and $BD = DF$.

Now sum of sqq. on AD, DB

= sq. on AD together with sq. on DF

= sq. on AF I. 47.

= sq. on AE together with sq. on EF I. 47.

= sqq. on AC, EC together with sqq. on EG, GF I. 47.

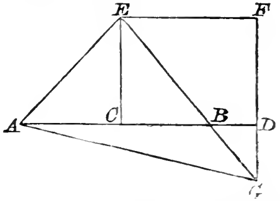
= twice sq. on AC together with twice sq. on GF

= twice sq. on AC together with twice sq. on CD.

Q. E. D.

Proposition X. Theorem.

If a straight line be bisected and produced to any point, the square on the whole line thus produced and the square on the part of it produced are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.



Let the st. line AB be bisected in C and produced to D.

Then must

sum of sqq. on AD, BD = twice sum of sqq. on AC, CD.

Draw $CE \perp$ to AB, and make $CE = AC$.

Join EA, EB and draw $EF \parallel$ to AD and $DF \parallel$ to CE.

Then $\because$ $\angle$ s FEB, EFD are together less than two rt. $\angle$ s,

$\therefore$ EB and FD will meet if produced towards B, D in some pt. G.

Join AG.

Then $\because$ $\angle\, ACE$ is a rt. $\angle$,

$\therefore$ $\angle$ s EAC, AEC together = a rt. $\angle$,

and $\because$ $\angle\, EAC = \angle\, AEC$, I. A

$\therefore$ $\angle\, AEC$ = half a rt. $\angle$.

So also $\angle$ s BEC, EBC each = half a rt. $\angle$.

$\therefore$ $\angle\, AEB$ is a rt. $\angle$.

Also $\angle\, DBG$, which = $\angle\, EBC$, is half a rt. $\angle$,

and $\therefore$ $\angle\, BGD$ is half a rt. $\angle$;

$\therefore$ $BD = DG$. I. B. Cor.

Again, $\because$ $\angle\, FGE$ = half a rt. $\angle$, and $\angle\, EFG$ is a rt. $\angle$, I. 34.

$\therefore$ $\angle\, FEG$ = half a rt. $\angle$, and $EF = FG$. I. B. Cor.

Then sum of sqq. on AD, DB

= sum of sqq. on AD, DG

= sq. on AG I. 47.

= sq. on AE together with sq. on EG I. 47.

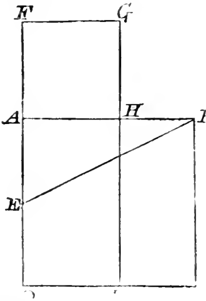
= sqq. on AC, EC together with sqq. on EF, FG I. 47.

= twice sq. on AC together with twice sq. on EF

= twice sq. on AC together with twice sq. on CD. Q. E. D.

PROPOSITION XI. PROBLEM.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.



Let AB be the given st. line.

On AB descr. the sq. $ADCB$. I. 46.

Bisect AD in E and join EB. I. 10.

Produce DA to F, making $EF=EB$.

On AF descr. the sq. $AFGH$. I. 46.

Then AB is divided in H so that rect. AB, BH = sq. on AH.

Produce GH to K.

Then $\because$ DA is bisected in E and produced to F,

$\therefore$ rect. DF, FA together with sq. on AE

=sq. on EF II. 6.

=sq. on EB, $\because$ $EB=EF$,

=sum of sqq. on AB, AE. I. 47.

Take from each the square on AE.

Then rect. DF, FA = sq. on AB. Ax. 3.

Now FK = rect. DF, FA, $\because$ $FG=FA$.

$\therefore$ $FK=AC$.

Take from each the common part AK.

Then $FH=HC$;

that is, sq. on AH = rect. AB, BH, $\because$ $BC=AB$.

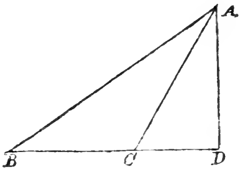
Thus AB is divided in H as was reqd.

Q. E. F.

Ex. Shew that the squares on the whole line and one of the parts are equal to three times the square on the other part.

Proposition XII. Theorem.

In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side, upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.



Let ABC be an obtuse-angled $\triangle$, having $\angle ACB$ obtuse.

From A draw $AD \perp$ to BC produced.

Then must sq. on AB be greater than sum of sqq. on BC, CA by twice rect. BC, CD.

For since BD is divided into two parts in C,
sq. on BD=sum of sqq. on BC, CD, and twice rect. BC, CD. II. 4.

Add to each sq. on DA: then
sum of sqq. on BD, DA=sum of sqq. on BC, CD, DA and twice rect. BC, CD.

Now sqq. on BD, DA=sq. on AB, I. 47.
and sqq. on CD, DA=sq. on CA; I. 47.

$\therefore$ sq. on AB=sum of sqq. on BC, CA and twice rect. BC, CD.

$\therefore$ sq. on AB is greater than sum of sqq. on BC, CA by twice rect. BC, CD.

Q. E. D.

Ex. The squares on the diagonals of a trapezium are together equal to the squares on its two sides, which are not parallel, and twice the rectangle contained by the sides, which are parallel.

Proposition XIII. Theorem.

In every triangle, the square on the side subtending any of the acute angles is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides and the straight line intercepted between the perpendicular, let fall upon it from the opposite angle, and the acute angle.

Fig. 1.

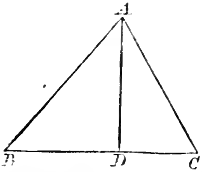
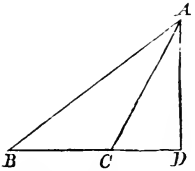


Fig. 2.



Let ABC be any $\triangle$, having the $\angle ABC$ acute.

From A draw $AD \perp$ to BC or BC produced.

Then must sq. on AC be less than the sum of sqq. on AB, BC, by twice rect. BC, BD.

For in Fig. 1 BC is divided into two parts in D,
and in Fig. 2 BD is divided into two parts in C;

$\therefore$ in both cases

sum of sqq. on BC, BD=sum of twice rect. BC, BD and sq. on CD. II. 7.

Add to each the sq. on DA, then

sum of sqq. on BC, BD, DA=sum of twice rect. BC, BD and sqq. on CD, DA;

$\therefore$ sum of sqq. on BC, AB=sum of twice rect. BC, BD and sq. on AC; I. 47.

$\therefore$ sq. on AC is less than sum of sqq. on AB, BC by twice rect. BC, BD.

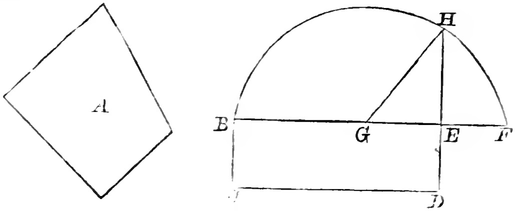
The case, in which the perpendicular AD coincides with AC, needs no proof.

Q. E. D.

Ex. Prove that the sum of the squares on any two sides of a triangle is equal to twice the sum of the squares on half the base and on the line joining the vertical angle with the middle point of the base.

Proposition XIV. Problem.

To describe a square that shall be equal to a given rectilinear figure.



Let A be the given rectil. figure.

It is reqd. to describe a square that shall $=A$.

Describe the rectangular ▱ $BCDE=A$. I. 45.

Then if $BE=ED$ the ▱ $BCDE$ is a square,

and what was reqd. is done.

But if BE be not $=ED$, produce BE to F, so that $EF=ED$.

Bisect BF in G; and with centre G and distance GB,

describe the semicircle BHF.

Produce DE to H and join GH.

Then, $\because$ BF is divided equally in G and unequally in E,

$\therefore$ rect. BE, EF together with sq. on GE

$=$sq. on GF II. 5.

$=$sq. on GH

$=$sum of sqq. on EH, GE. I. 47.

Take from each the square on GE.

Then rect. BE, $EF=$sq. on EH.

But rect. BE, $EF=BD$, $\because$ $EF=ED$;

$\therefore$ sq. on $EH=BD$;

$\therefore$ sq. on $EH=$rectil. figure A.

Q. E. F.

Miscellaneous Exercises on Book II.

1. In a triangle, whose vertical angle is a right angle, a straight line is drawn from the vertex perpendicular to the base; shew that the square on either of the sides adjacent to the right angle is equal to the rectangle contained by the base and the segment of it adjacent to that side.

2. The squares on the diagonals of a parallelogram are together equal to the squares on the four sides.

3. If $ABCD$ be any rectangle, and O any point either within or without the rectangle, shew that the sum of the squares on OA, OC is equal to the sum of the squares on OB, OD.

4. If either diagonal of a parallelogram be equal to one of the sides about the opposite angle of the figure, the square on it shall be less than the square on the other diameter, by twice the square on the other side about that opposite angle.

5. Produce a given straight line AB to C, so that the rectangle, contained by the sum and difference of AB and AC, may be equal to a given square.

6. Shew that the sum of the squares on the diagonals of any quadrilateral is less than the sum of the squares on the four sides, by four times the square on the line joining the middle points of the diagonals.

7. If the square on the perpendicular from the vertex of a triangle is equal to the rectangle, contained by the segments of the base, the vertical angle is a right angle.

8. If two straight lines be given, shew how to produce one of them so that the rectangle contained by it and the produced part may be equal to the square on the other.

9. If a straight line be divided into three parts, the square on the whole line is equal to the sum of the squares on the parts together with twice the rectangle contained by each two of the parts.

10. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.

11. If straight lines be drawn from each angle of a triangle to bisect the opposite sides, four times the sum of the squares on these lines is equal to three times the sum of the squares on the sides of the triangle.

12. CD is drawn perpendicular to AB, a side of the triangle ABC, in which $AC = AB$. Shew that the square on CD is equal to the square on BD together with twice the rectangle AD, DB.

13. The hypotenuse AB of a right-angled triangle ABC is trisected in the points D, E; prove that if CD, CE be joined, the sum of the squares on the sides of the triangle CDE is equal to two-thirds of the square on AB.

14. The square on the hypotenuse of an isosceles right-angled triangle is equal to four times the square on the perpendicular from the right angle on the hypotenuse.

15. Divide a given straight line into two parts, so that the rectangle contained by them shall be equal to the square described upon a straight line, which is less than half the line divided.

Note 6.—*On the Measurement of Areas.*

To measure a Magnitude, we fix upon some magnitude of the same kind to serve as a standard or unit; and then any magnitude of that kind is measured by the number of times it contains this unit, and this number is called the Measure of the quantity.

Suppose, for instance, we wish to measure a straight line AB. We take another straight line EF for our standard,

and then we say

if AB contain EF three times, the measure of AB is **3**,
if ....four..**4**,
if .... x .... x.

Next suppose we wish to measure two straight lines AB, CD by the same standard EF.

If AB contain EF m times
and CD .... n times,

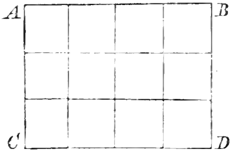
where m and n stand for numbers, whole or fractional, we say that AB and CD are *commensurable.*

But it may happen that we may be able to find a standard line EF, such that it is contained an exact number of times in AB; and yet there is no number, whole or fractional, which will express the number of times EF is contained in CD.

In such a case, where no unit-line can be found, such that it is contained an exact number of times in *each* of two lines AB, CD, these two lines are called *incommensurable.*

In the processes of Geometry we constantly meet with incommensurable magnitudes. Thus the side and diagonal of a square are incommensurables; and so are the diameter and circumference of a circle.

Next, suppose two lines AB, AC to be at right angles to each other and to be commensurable, so that AB contains four times a certain unit of linear measurement, which is contained by AC three times.



Divide AB, AC into four and three equal parts respectively, and draw lines through the points of division parallel to AC, AB respectively; then the rectangle $ACDB$ is divided into a number of equal squares, each constructed on a line equal to the unit of linear measurement.

If one of these squares be taken as the unit of area, the *measure* of the area of the rectangle $ACDB$ will be the number of these squares.

Now this number will evidently be the same as that obtained by multiplying the measure of AB by the measure of AC; that is, the measure of AB being 4 and the measure of AC 3, the measure of $ACDB$ is 4×3 or 12. (Algebra, Art. 38.)

And *generally*, if the measures of two adjacent sides of a rectangle, supposed to be commensurable, be a and b, then the measure of the rectangle will be ab. (Algebra, Art. 39.)

If all lines were commensurable, then, whatever might be the length of two adjacent sides of a rectangle, we might select the unit of length, so that the measures of the two sides should be whole numbers; and then we might apply the processes of Algebra to establish many Propositions in Geometry by simpler methods than those adopted by Euclid.

Take, for example, the theorem in Book II. Prop. IV.

If all lines were commensurable we might proceed thus.—

Let the measure of AC be x,

.... of CB ... y,

Then the measure of AB is $x+y$.

Now $$(x+y)^2 = x^2 + y^2 + 2xy,$$

which proves the theorem.

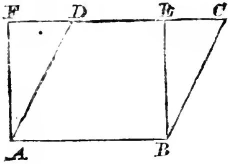
But, inasmuch as all lines are not commensurable, we have in Geometry to treat of *magnitudes* and not of *measures*: that is, when we use the symbol A to represent a line (as in I. 22), A stands for the line itself and not, as in Algebra, for the number of units of length contained by the line.

The method, adopted by Euclid in Book II. to explain the relations between the rectangles contained by certain lines, is more exact than any method founded upon Algebraical principles can be; because his method applies not merely to the case in which the sides of a rectangle are commensurable, but also to the case in which they are incommensurable.

The student is now in a position to understand the practical application of the theory of Equivalence of Areas, of which the foundation is the 35th Proposition of Book I. We shall give a few examples of the use made of this theory in Mensuration.

Area of a Parallelogram.

The area of a parallelogram $ABCD$ is equal to the area of the rectangle $ABEF$ on the same base AB and between the same parallels AB, FC.



Now BE is the altitude of the parallelogram $ABCD$ if AB be taken as the base.

Hence area of ▱ $ABCD$ = rect. AB, BE.

If then the measure of the base be denoted by b,

and .... altitude .... h,

the measure of the area of the ▱ will be denoted by bh

That is, when the base and altitude are commensurable,

measure of area = measure of base into measure of altitude.

Area of a Triangle.

If from one of the angular points A of a triangle ABC, a perpendicular AD be drawn to BC, Fig. 1, or to BC produced, Fig. 2,

Fig. 1.

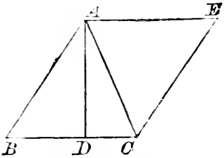
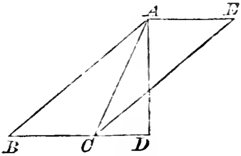


Fig. 2.



and if, in both cases, a parallelogram $ABCE$ be completed of which AB, BC are adjacent sides,

area of $\triangle\, ABC$ = half of area of $\square\, ABCE$.

Now if the measure of BC be b,

and .... AD... h,

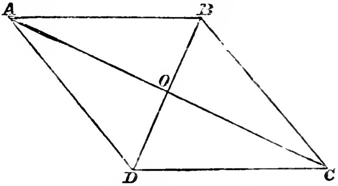
measure of area of $\square\, ABCE$ is bh;

$\therefore$ measure of area of $\triangle\, ABC$ is $\frac{bh}{2}$.

Area of a Rhombus.

Let $ABCD$ be the given rhombus.

Draw the diagonals AC and BD, cutting one another in O.



It is easy to prove that AC and BD bisect each other at right angles.

Then if the measure of AC be x,

and .... BD ... y,

measure of area of rhombus = twice measure of $\triangle\, ACD$.

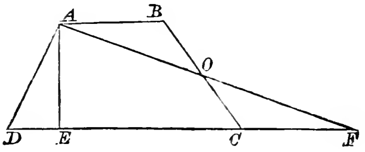
$$= \text{twice } \frac{xy}{4}$$

$$= \frac{xy}{2}.$$

Area of a Trapezium.

Let $ABCD$ be the given trapezium, having the sides AB, CD parallel.

Draw AE at right angles to CD.



Produce DC to F, making $CF=AB$.

Join AF, cutting BC in O.

Then in $\triangle$s AOB, COF,

$\because \angle BAO = \angle CFO$, and $\angle AOB = \angle FOC$, and $AB=CF$;

$\therefore \triangle COF = \triangle AOB$. I. 26.

Hence trapezium $ABCD = \triangle ADF$.

Now suppose the measures of AB, CD, AE to be m, n, p respectively;

$\therefore$ measure of $DF = m+n$, $\because CF=AB$.

Then measure of area of trapezium

$= \frac{1}{2}$ (measure of $DF \times$ measure of AE)

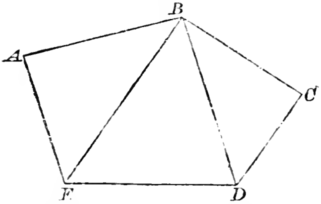
$= \frac{1}{2}(m+n) \times p$.

That is, the measure of the area of a trapezium is found by multiplying half the measure of the sum of the parallel sides by the measure of the perpendicular distance between the parallel sides.

Area of an Irregular Polygon.

There are three methods of finding the area of an irregular polygon, which we shall here briefly notice.

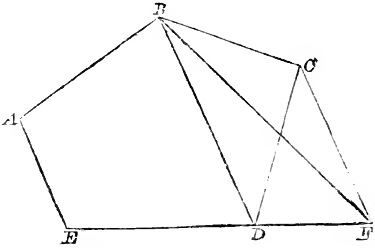
I. *The polygon may be divided into triangles*, and the area of each of these triangles be found separately.



Thus the area of the irregular polygon $ABCDE$ is equal to the sum of the areas of the triangles ABE, EBD, DBC.

II. *The polygon may be converted into a single triangle of equal area.*

If $ABCDE$ be a pentagon, we can convert it into an equivalent quadrilateral by the following process:



Join BD and draw CF parallel to BD, meeting ED produced in F, and join BF.

Then will quadrilateral $ABFE$=pentagon $ABCDE$.

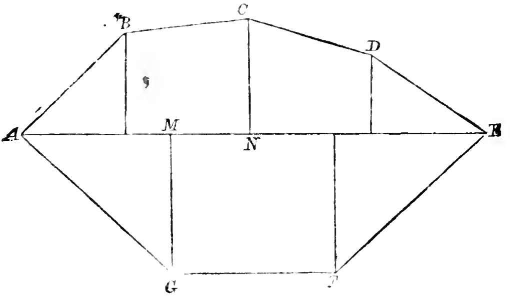
For $\triangle BDF = \triangle BCD$, on same base BD and between same parallels.

If, then, from the pentagon we remove $\triangle BCD$, and add $\triangle BDF$ to the remainder, we obtain a quadrilateral $ABFE$ equivalent to the pentagon $ABCDE$.

The quadrilateral may then, by a similar process, be converted into an equivalent triangle, and thus a polygon of any number of sides may be gradually converted into an equivalent triangle.

The area of this triangle may then be found.

III. The third method is chiefly employed in practice by Surveyors



Let $ABCDEFG$ be an irregular polygon.

Draw AE, the longest diagonal, and drop perpendiculars on AE from the other angular points of the polygon.

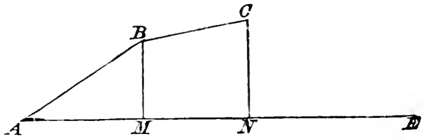
The polygon is thus divided into figures which are either right-angled triangles, rectangles, or trapeziums ; and the areas of each of these figures may be readily calculated.

NOTE 7. *On Projections.*

The projection of a *point* B, on a straight line of unlimited length AE, is the point M at the foot of the perpendicular dropped from B on AE.

The projection of a *straight line* BC, on a straight line of unlimited length AE, is MN,—the part of AE intercepted between perpendiculars drawn from B and C.

When two lines, as AB and AE, form an angle, the projection of AB on AE is AM.



We might employ the term projection with advantage to shorten and make clearer the enunciations of Props. XII. and XIII. of Book II.

Thus the enunciation of Prop. XII. might be :—

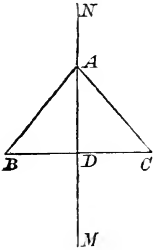
"In oblique-angled triangles, the square on the side subtending the obtuse angle is greater than the squares on the sides containing that angle, by twice the rectangle contained by one of these sides and the projection of the other on it."

The enunciation of Prop. XIII. might be altered in a similar manner.

Note 8. *On Loci.*

Suppose we have to determine the position of a point, which is equidistant from the extremities of a given straight line BC.

There is an infinite number of points satisfying this condition, for the vertex of any isosceles triangle, described on BC as its base, is equidistant from B and C.



Let ABC be *one* of the isosceles triangles described on BC.

If BC be bisected in D, MN, a perpendicular to BC drawn through D, will pass through A.

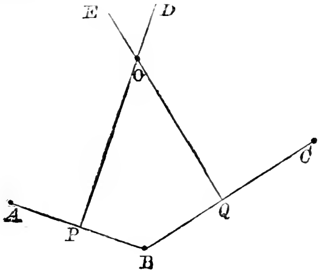
It is easy to shew that any point in MN, or MN produced in either direction, is equidistant from B and C.

It may also be proved that no point out of MN is equidistant from B and C.

The line MN is called the Locus of all the points, infinite in number, which are equidistant from B and C.

Def. In plane Geometry *Locus* is the name given to a line, straight or curved, all of whose points satisfy a certain geometrical condition (or have a common property), to the exclusion of all other points.

Next, suppose we have to determine the position of a point, which is equidistant from three given points A, B, C, not in the same straight line.



If we join A and B, we know that all points equidistant from A and B lie in the line PD, which bisects AB at right angles.

If we join B and C, we know that all points equidistant from B and C lie in the line QE, which bisects BC at right angles.

Hence O, the point of intersection of PD and QE, is the only point equidistant from A, B and C.

PD is the Locus of points equidistant from A and B,

QE.. B and C,

and the Intersection of these Loci determines the point, which is equidistant from A, B and C.

Examples of Loci.

Find the loci of

(1) Points at a given distance from a given point.

(2) Points at a given distance from a given straight line.

(3) The middle points of straight lines drawn from a given point to a given straight line.

(4) Points equidistant from the arms of an angle.

(5) Points equidistant from a given circle.

(6) Points equally distant from two straight lines which intersect.

Note 9. *On the Methods employed in the solution of Problems.*

In the solution of Geometrical Exercises, certain methods may be applied with success to particular classes of questions.

We propose to make a few remarks on these methods, so far as they are applicable to the first two books of Euclid's Elements.

The Method of Synthesis.

In the Exercises, attached to the Propositions in the preceding pages, the construction of the diagram, necessary for the solution of each question, has usually been fully described, or sufficiently suggested.

The student has in most cases been required simply to apply the geometrical fact, proved in the Proposition preceding the exercise, in order to arrive at the conclusion demanded in the question.

This way of proceeding is called Synthesis (σύνθεσις = composition), because in it we proceed by a regular chain of reasoning from what is *given* to what is *sought.* This being the method employed by Euclid throughout the Elements, we have no need to exemplify it here.

The Method of Analysis.

The solution of many Problems is rendered more easy *by supposing the problem solved and the diagram constructed.* It is then often possible to observe relations between lines, angles and figures in the diagram, which are suggestive of the steps by which the necessary construction might have been effected.

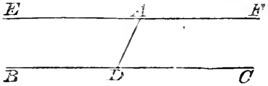
This is called the Method of Analysis (ἀνάλυσις = resolution). It is a method of discovering truth by reasoning concerning things unknown or propositions merely supposed, as if the one were given or the other were really true. The process can best be explained by the following examples.

Our first example of the Analytical process shall be the 31st Proposition of Euclid's First Book.

Ex. 1. *To draw a straight line through a given point parallel to a given straight line.*

Let A be the given point, and BC be the given straight line.

Suppose the problem to be effected, and EF to be the straight line required.



Now we know that any straight line AD drawn from A to meet BC makes equal angles with EF and BC. (I. 29.)

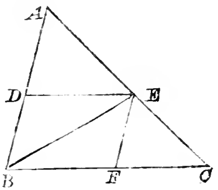
This is a fact from which we can work backward, and arrive at the steps necessary for the solution of the problem ; thus :

Take any point D in BC, join AD, make $\angle EAD = \angle ADC$, and produce EA to F: then EF must be parallel to BC.

Ex. 2. *To inscribe in a triangle a rhombus, having one of its angles coincident with an angle of the triangle.*

Let ABC be the given triangle.

Suppose the problem to be effected, and $DBFE$ to be the rhombus.



Then if EB be joined, $\angle DBE = \angle FBE$.

This is a fact from which we can work backward, and deduce the necessary construction ; thus :

Bisect $\angle ABC$ by the straight line BE, meeting AC in E.

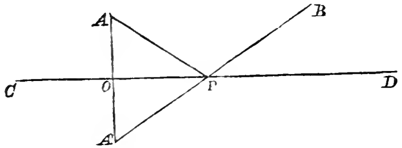
Draw ED and EF parallel to BC and AB respectively.

Then $DBFE$ is the rhombus required. (See Ex. 4, p. 59.)

Ex. 3. *To determine the point in a given straight line, at which straight lines, drawn from two given points, on the same side of the given line, make equal angles with it.*

Let CD be the given line, and A and B the given points.

Suppose the problem to be effected, and P to be the point required.



We then reason thus :

If BP were produced to some point A',

$\angle CPA'$, being $= \angle BPD$, will be $= \angle APC$.

Again, if PA' be made equal to PA,

AA' will be bisected by CP at right angles.

This is a fact from which we can work backward, and find the steps necessary for the solution of the problem ; thus :

From A draw $AO \perp$ to CD.

Produce AO to A', making $OA' = OA$.

Join BA', cutting CD in P.

Then P is the point required.

NOTE 10. *On Symmetry.*

The problem, which we have just been considering, suggests the following remarks :

If two points, A and A', be so situated with respect to a straight line CD, that CD bisects at right angles the straight line joining A and A', then A and A' are said to be *symmetrical* with regard to CD.

The importance of symmetrical relations, as suggestive of methods for the solution of problems, cannot be fully shewn

to a learner, who is unacquainted with the properties of the circle. The following example, however, will illustrate this part of the subject sufficiently for our purpose at present.

Find a point in a given straight line, such that the sum of its distances from two fixed points on the same side of the line is a minimum, that is, less than the sum of the distances of any other point in the line from the fixed points.

Taking the diagram of the last example, suppose CD to be the given line, and A, B the given points.

Now if A and A' be symmetrical with respect to CD, we know that *every* point in CD is equally distant from A and A'. (See Note 8, p. 103.)

Hence the sum of the distances of any point in CD from A and B is equal to the sum of the distances of that point from A' and B.

But the sum of the distances of a point in CD from A' and B is the least possible when it lies in the straight line joining A' and B.

Hence the point P, *determined as in the last example*, is the point required.

Note. Propositions IX., X., XI., XII. of Book I. give good examples of symmetrical constructions.

Note 11. *Euclid's Proof of I. 5.*

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall be equal.

Let ABC be an isosceles $\triangle$, having $AB = AC$

Produce AB, AC to D and E.

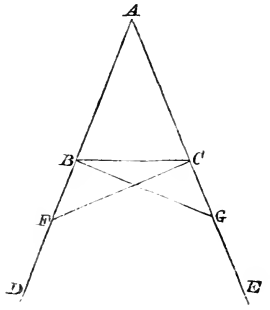
Then must $\angle\, ABC = \angle\, ACB$,

and $\angle\, DBC = \angle\, ECB$.

In BD take any pt. F.

From AE cut off $AG = AF$.

Join FC and GB.



Then in $\triangle$s AFC, AGB,

$\because FA = GA$, and $AC = AB$, and $\angle FAC = \angle GAB$,

$\therefore FC = GB$, and $\angle AFC = \angle AGB$, and $\angle ACF = \angle ABG$. I. 4.

Again, $\because AF = AG$,

of which the parts AB, AC are equal,

$\therefore$ remainder BF = remainder CG. Ax. 3.

Then in $\triangle$s BFC, CGB,

$\because BF = CG$, and $FC = GB$, and $\angle BFC = \angle CGB$,

$\therefore \angle FBC = \angle GCB$, and $\angle BCF = \angle CBG$, I. 4.

Now it has been proved that $\angle ACF = \angle ABG$,

of which the parts $\angle BCF$ and $\angle CBG$ are equal;

$\therefore$ remaining $\angle ACB$ = remaining $\angle ABC$. Ax. 3.

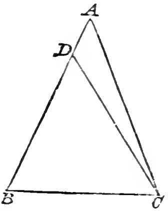
Also it has been proved that $\angle FBC = \angle GCB$,

that is, $\angle DBC = \angle ECB$.

Q. E. D.

NOTE 12. *Euclid's Proof of I. 6.*

If two angles of a triangle be equal to one another, the sides also, which subtend the equal angles, shall be equal to one another.



In $\triangle ABC$ let $\angle ACB = \angle ABC$.

Then must $AB = AC$.

For if not, AB is either greater or less than AC.

Suppose AB to be greater than AC.

From AB cut off $BD = AC$, and join DC.

Then in $\triangle$s DBC, ACB,

$\because DB = AC$, and BC is common, and $\angle DBC = \angle ACB$,

$\therefore \triangle DBC = \triangle ACB$; I. 4.

that is, the less = the greater; which is absurd.

$\therefore AB$ is not greater than AC.

Similarly it may be shewn that AB is not less than AC;

$\therefore AB = AC$.

Q. E. D.

NOTE 13. *Euclid's Proof of I. 7.*

Upon the same base and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and their sides which are terminated in the other extremity of the base equal also.

If it be possible, on the same base AB, and on the same side of it, let there be two $\triangle$s ACB, ADB, such that $AC = AD$, and also $BC = BD$.

Join CD.

First, when the vertex of each of the $\triangle$s is *outside* the other $\triangle$ (Fig. 1.);

Fig 1.

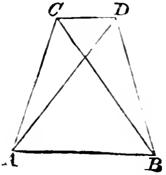
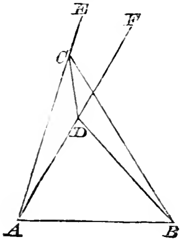


Fig. 2.



$\because AD = AC,$

$\therefore \angle ACD = \angle ADC.$ I. 5.

But $\angle ACD$ is greater than $\angle BCD$;

$\therefore \angle ADC$ is greater than $\angle BCD$;

much more is $\angle BDC$ greater than $\angle BCD.$

Again, $\because BC = BD,$

$\therefore \angle BDC = \angle BCD,$

that is, $\angle BDC$ is both equal to and greater than $\angle BCD$; which is absurd.

Secondly, when the vertex D of one of the $\triangle$s falls *within* the other $\triangle$ (Fig. 2);

Produce AC and AD to E and F

Then $\because AC = AD.$

$\therefore \angle ECD = \angle FDC.$ I. 5.

But $\angle ECD$ is greater than $\angle BCD$;

$\therefore \angle FDC$ is greater than $\angle BCD$;

much more is $\angle BDC$ greater than $\angle BCD.$

Again, $\because BC = BD.$

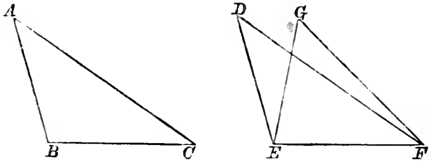
$\therefore \angle BDC = \angle BCD$;

that is, $\angle BDC$ is both equal to and greater than $\angle BCD$; which is absurd.

Lastly, when the vertex D of one of the $\triangle$s falls on a side BC of the other, it is plain that BC and BD cannot be equal. Q. E. D.

Note 14. *Euclid's Proof of I. 8.*

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one must be equal to the angle contained by the two sides of the other.



Let the sides of the $\triangle$s ABC, DEF be equal, each to each, that is, $AB=DE$, $AC=DF$ and $BC=EF$.

Then must $\angle BAC = \angle EDF$.

Apply the $\triangle ABC$ to the $\triangle DEF$.

so that pt. B is on pt. E, and BC on EF.

Then $\because BC=EF$,

$\therefore C$ will coincide with F,

and BC will coincide with EF.

Then AB and AC must coincide with DE and DF.

For if AB and AC have a different position, as GE, GF, then upon the same base and upon the same side of it there can be two $\triangle$s, which have their sides which are terminated in one extremity of the base equal, and their sides which are terminated in the other extremity of the base also equal: which is impossible. I. 7.

$\therefore$ since base BC coincides with base EF,

AB must coincide with DE, and AC with DF;

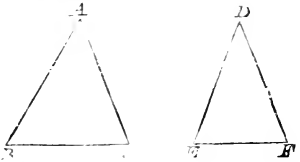
$\therefore \angle BAC$ coincides with and is equal to $\angle EDF$.

Q. E. D.

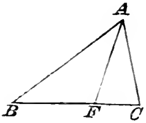
Note 15. *Another Proof of I. 24.*

In the $\triangle$s ABC, DEF, let $AB=DE$ and $AC=DF$, and let $\angle BAC$ be greater than $\angle EDF$.

Then must BC be greater than EF.



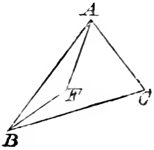
Apply the $\triangle$ DEF to the $\triangle$ ABC
so that DE coincides with AB.
Then $\because$ $\angle EDF$ is less than $\angle BAC$,
DF will fall between BA and AC,
and F will fall *on*, or *above*, or *below*, BC.



I. If F fall on BC,

BF is less than BC;

$\therefore$ EF is less than BC.



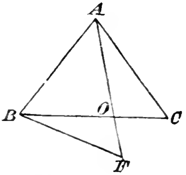
II. If F fall *above* BC,

BF, FA together are less than BC, CA,

and $FA=CA$;

$\therefore$ BF is less than BC;

$\therefore$ EF is less than BC.



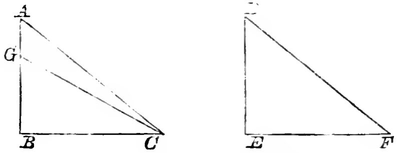
III. If F fall *below* BC.

let AF cut BC in O.

Then BO, OF together are greater than BF, I. 20.
and OC, AO .... AC; I. 20.
$\therefore$ BC, AF .... BF, AC together,
and $AF=AC$,
$\therefore$ BC is greater than BF;
and $\therefore$ EF is less than BC. Q. E. D.

NOTE 16. *Euclid's Proof of I. 26.*

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, viz., either the sides adjacent to the equal angles, or the sides opposite to equal angles in each; then shall the other sides be equal, each to each; and also the third angle of the one to the third angle of the other.



In $\triangle$s ABC, DEF,

Let $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$;

and *first*,

Let the sides adjacent to the equal $\angle$s in each be equal, that is, let $BC = EF$.

Then must $AB = DE$, and $AC = DF$, and $\angle BAC = \angle EDF$.

For if AB be not $= DE$, one of them must be the greater.

Let AB be the greater, and make $GB = DE$, and join GC

Then in $\triangle$s GBC, DEF,

$\because GB = DE$, and $BC = EF$, and $\angle GBC = \angle DEF$,

$\therefore \angle GCB = \angle DFE$. I. 4.

But $\angle ACB = \angle DFE$ by hypothesis;

$\therefore \angle GCB = \angle ACB$;

that is, the less = the greater, which is impossible.

$\therefore AB$ is not greater than DE.

In the same way it may be shewn that AB is not less than DE;

$\therefore AB = DE$.

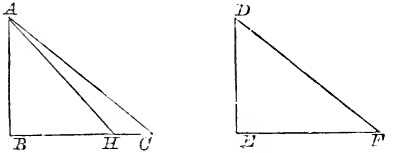
Then in $\triangle$s ABC, DEF,

$\because AB = DE$, and $BC = EF$, and $\angle ABC = \angle DEF$,

$\therefore AC = DF$, and $\angle BAC = \angle EDF$. I. 4.

Next, let the sides which are opposite to equal angles in each triangle be equal, viz., $AB=DE$.

Then must $AC=DF$, *and* $BC=EF$, *and* $\angle BAC = \angle EDF$.



For if BC be not $=EF$, let BC be the greater, and make $BH=EF$, and join AH.

Then in $\triangle$s ABH, DEF,

$\because AB=DE$, and $BH=EF$, and $\angle ABH = \angle DEF$,

$\therefore \angle AHB = \angle DFE$. I. 4.

But $\angle ACB = \angle DFE$, by hypothesis,

$\therefore \angle AHB = \angle ACB$;

that is, the exterior $\angle$ of $\triangle AHC$ is equal to the interior and opposite $\angle ACB$, which is impossible.

$\therefore BC$ is not greater than EF.

In the same way it may be shewn that BC is not less than EF;

$\therefore BC=EF$.

Then in $\triangle$s ABC, DEF,

$\because AB=DE$, and $BC=EF$, and $\angle ABC = \angle DEF$,

$\therefore AC=DF$, and $\angle BAC = \angle EDF$. I. 4.

Q. E. D.

Miscellaneous Exercises on Books I. and II.

1. AB and CD are equal straight lines, bisecting one another at right angles. Shew that $ACBD$ is a square.

2. From a point in the side of a parallelogram draw a line dividing the parallelogram into two equal parts.

3. In the triangle FDC, if FCD be a right angle, and angle FDC be double of angle CFD, shew that FD is double of DC.

4. If ABC be an equilateral triangle, and AD, BE be perpendiculars to the opposite sides intersecting in F; shew that the square on AB is equal to three times the square on AF.

5. Describe a rhombus, which shall be equal to a given triangle, and have each of its sides equal to one side of the triangle.

6. From a given point, outside a given straight line, draw a line making with the given line an angle equal to a given rectilineal angle.

7. If two straight lines be drawn from two given points to meet in a given straight line, shew that the sum of these lines is the least possible, when they make equal angles with the given line.

8. $ABCD$ is a parallelogram, whose diagonals AC, BD intersect in O; shew that if the parallelograms $AOBP$, $DOCQ$ be completed, the straight line joining P and Q passes through O.

9. $ABCD$, $EBCF$ are two parallelograms on the same base BC, and so situated that CF passes through A. Join DF, and produce it to meet BE produced in K; join FB, and prove that the triangle FAB equals the triangle FEK.

10. The alternate sides of a polygon are produced to meet; shew that all the angles at their points of intersection together with four right angles are equal to all the interior angles of the polygon.

11. Shew that the perimeter of a rectangle is always greater than that of the square equal to the rectangle.

12. Shew that the opposite sides of an equiangular hexagon are parallel, though they be not equal.

13. If two equal straight lines intersect each other anywhere at right angles, shew that the area of the quadrilateral formed by joining their extremities is invariable, and equal to one-half the square on either line.

14. Two triangles ACB, ADB are constructed on the same side of the same base AB. Shew that if $AC=BD$ and $AD=BC$, then CD is parallel to AB; but if $AC=BC$ and $AD=BD$, then CD is perpendicular to AB.

15. AB is the hypotenuse of a right-angled triangle ABC: find a point D in AB, such that DB may be equal to the perpendicular from D on AC.

16. Find the locus of the vertices of triangles of equal area on the same base, and on the same side of it.

17. Shew that the perimeter of an isosceles triangle is less than that of any triangle of equal area on the same base.

18. If each of the equal angles of an isosceles triangle be equal to one-fourth the vertical angle, and from one of them a perpendicular be drawn to the base, meeting the opposite side produced, then will the part produced, the perpendicular, and the remaining side, form an equilateral triangle.

19. If a straight line terminated by the sides of a triangle be bisected, shew that no other line terminated by the same two sides can be bisected in the same point.

20. Shew how to bisect a given quadrilateral by a straight line drawn from one of its angles.

21. Given the lengths of the two diagonals of a rhombus, construct it.

22. $ABCD$ is a quadrilateral figure: construct a triangle whose base shall be in the line AB, such that its altitude shall be equal to a given line, and its area equal to that of the quadrilateral.

23. If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides, their sum will be equal to the perpendicular from either extremity of the base upon the opposite side.

24. If ABC be a triangle, in which C is a right angle, and DE be drawn from a point D in AC at right angles to AB, prove that the rectangles AB, AE and AC, AD are equal.

25. A line is drawn bisecting parallelogram $ABCD$, and meeting AD, BC in E and F: shew that the triangles EBF, CED are equal.

26. Upon the hypotenuse BC and the sides CA, AB of a right-angled triangle ABC, squares $BDEC$, AF and AG are described: shew that the squares on DG and EF are together equal to five times the square on BC.

27. If from the vertical angle of a triangle three straight lines be drawn, one bisecting the angle, the second bisecting the base, and the third perpendicular to the base, shew that the first lies, both in position and magnitude, between the other two.

28. If ABC be a triangle, whose angle A is a right angle, and BE, CF be drawn bisecting the opposite sides respectively, shew that four times the sum of the squares on BE and CF is equal to five times the square on BC.

29. Let ACB, ADB be two right-angled triangles having a common hypotenuse AB. Join CD and on CD produced both ways draw perpendiculars AE, BF. Shew that the sum of the squares on CE and CF is equal to the sum of the squares on DE and DF.

30. In the base AC of a triangle take any point D: bisect AD, DC, AB, BC at the points E, F, G, H respectively. Shew that EG is equal and parallel to FH.

31. If AD be drawn from the vertex of an isosceles triangle ABC to a point D in the base, shew that the rectangle BD, DC is equal to the difference between the squares on AB and AD.

32. If in the sides of a square four points be taken at equal distances from the four angular points taken in order, the figure contained by the straight lines, which join them, shall also be a square.

33. If the sides of an equilateral and equiangular pentagon be produced to meet, shew that the sum of the angles at the points of meeting is equal to two right angles.

34. Describe a square that shall be equal to the difference between two given and unequal squares.

35. $ABCD$, $AECF$ are two parallelograms, EA, AD being in a straight line. Let FG, drawn parallel to AC, meet BA produced in G. Then the triangle ABE equals the triangle ADG.

36. From AC, the diagonal of a square $ABCD$, cut off AE equal to one-fourth of AC, and join BE, DE. Shew that the figure $BADE$ is equal to twice the square on AE.

37. If ABC be a triangle, with the angles at B and C each double of the angle at A, prove that the square on AB is equal to the square on BC together with the rectangle AB, BC.

38. If two sides of a quadrilateral be parallel, the triangle contained by either of the other sides and the two straight lines drawn from its extremities to the middle point of the opposite side is half the quadrilateral.

39. Describe a parallelogram equal to and equiangular with a given parallelogram, and having a given altitude.

40. If the sides of a triangle taken in order be produced to twice their original lengths, and the outer extremities be joined, the triangle so formed will be seven times the original triangle.

41. If one of the acute angles of a right-angled isosceles triangle be bisected, the opposite side will be divided by the bisecting line into two parts, such that the square on one will be double of the square on the other.

42. ABC is a triangle, right-angled at B, and BD is drawn perpendicular to the base, and is produced to E until ECB is a right angle; prove that the square on BC is equal to the sum of the rectangles AD, DC and BD, DE.

43. Shew that the sum of the squares on two unequal lines is greater than twice the rectangle contained by the lines.

44. From a given isosceles triangle cut off a trapezium, having the base of the triangle for one of its parallel sides, and having the other three sides equal.

45. If any number of parallelograms be constructed having their sides of given length, shew that the sum of the squares on the diagonals of each will be the same.

46. $ABCD$ is a right-angled parallelogram, and AB is double of BC; on AB an equilateral triangle is constructed: shew that its area will be less than that of the parallelogram.

47. A point O is taken within a triangle ABC, such that the angles BOC, COA, AOB are equal; prove that the squares on BC, CA, AB are together equal to the rectangles contained by OB, OC; OC, OA; OA, OB; and twice the sum of the squares on OA, OB, OC.

48. If the sides of an equilateral and equiangular hexagon be produced to meet, the angles formed by these lines are together equal to four right angles.

49. ABC is a triangle right-angled at A; in the hypotenuse two points D, E are taken such that $BD=BA$ and $CE=CA$; shew that the square on DE is equal to twice the rectangle contained by BE, CD.

50. Given one side of a rectangle which is equal in area to a given square, find the other side.

51. AB, AC are the two equal sides of an isosceles triangle; from B, BD is drawn perpendicular to AC, meeting it in D; shew that the square on BD is greater than the square on CD by twice the rectangle AD, CD.

APPENDIX.

EXAMINATION PAPERS IN EUCLID

SET TO CANDIDATES FOR

First and Second Class Provincial Certificates,

AND TO STUDENTS MATRICULATING IN THE

UNIVERSITY OF TORONTO.

SECOND CLASS PROVINCIAL CERTIFICATES, 1871.

TIME—TWO HOURS AND A HALF.

1. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides, equal to them, of the other.
2. Triangles upon the same base, and between the same parallels, are equal to one another.
3. If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.
4. If a straight line be divided into two equal, and also into two unequal, parts, the squares on the two unequal parts are together double of the square on half the line, and of the square on the line between the points of section.
5. If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.
6. Bisect a parallelogram by a straight line drawn from a point in one of its sides.
7. Let A B C be a triangle, and let B D be a straight line drawn to D, a point in A C between A and C, then, if A B be greater than A C, the excess of A B above A C is less than that of B D above D C.
8. In a triangle A B C, A D being drawn perpendicular to the straight line B D which bisects the angle B, show that a line drawn from D parallel to B C will bisect A C.

NOTE.—The percentage of marks requisite, in order that a candidate may be ranked of a particular grade, will be taken on the value of the above paper, omitting question 8.

SECOND CLASS PROVINCIAL CERTIFICATES, 1872.

TIME—$2\frac{3}{4}$ HOURS.

1. Define a *straight line*, a *plane rectilineal angle*, a *right angle*, a *Gnomon*. Enunciate Euclid's Postulates.
2. If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.
3. If two triangles have two angles of the one equal to angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each; then shall the other sides be equal, each to each; and also the third angle of the one equal to the third angle of the other. *(Take the case in which the assumed equal sides are those opposite to equal angles.)*
4. In every triangle, the square on the side subtending an acute angle is less than the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle, and acute angle. *(Take the case where the perpendicular falls within the triangle.)*
5. If a straight line be divided into any two parts, the squares on the whole line, and one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.
6. Prove that, if a straight line AD be drawn from A, one of the angles of a triangle ABC, to D, the middle point of the opposite side BC, BA $\times$ AC is greater than 2 AD.
7. Let the equilateral triangle ABC, and triangle ADB, in which the angle ABD is a right angle, be on the same base AB, and between the same parallels AB and CD. Prove that $4\ AD^2 = 7\ AB^2$
8. From D, a point in AB, a side of the triangle ABC, it is required to draw a straight line DE, cutting BC in E, and AC produced in F, so that DE may be equal to EF.

SECOND CLASS PROVINCIAL CERTIFICATES, 1873.

TIME—TWO HOURS AND A HALF.

NOTE.—Candidates who take only Book I, will confine themselves to the first eight questions; those who take Books I and II, will omit the first two questions.

1. If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.
2. If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.
3. The opposite side, and angles of a parallelogram, are equal to one another.
4. The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.
5. To describe a square on a given straight line.
6. Let A B C D be a quadrilateral figure whose opposite angles A B C and A D C are right angles. Prove that, if A B be equal to A D, C B and C D shall also be equal to one another.
7. If A B C D be a quadrilateral figure, having the side A B parallel to the side C D, the straight line which joins the middle points of A B and D C shall divide the quadrilateral into two equal parts.
8. The straight line, which joins the middle points of two sides of a triangle, is parallel to the base.
9. If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the parts.
10. In an obtuse angled triangle, is the sum of the sides containing the obtuse angle greater or less than the square of the side opposite to the obtuse angle? And, by how much? Prove the proposition.

SECOND CLASS PROVINCIAL CERTIFICATES, 1874.

TIME—TWO HOURS AND THREE-QUARTERS.

Note.—Candidates who take only Book I. will confine themselves to the first 7 questions. Those who take Books I. and II. will omit questions 1, 2, and 3.

1. When is one straight line said to be *perpendicular* to another.

   To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.
2. If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.
3. If two triangles have two angles of the one equal to two angles of the other, each to each; and one side equal to one side, namely, sides which are opposite to

equal angles in each; then shall the other sides be equal, each to each.

4. What are *parallel straight lines?*
If a straight line, falling on two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.

5. What is a *parallelogram?*
Parallelograms on equal bases, and between the same parallels, are equal to one another.

6. If two isosceles triangles be on the same base, and on the same side of it, the straight line which joins their vertices, will, if produced, cut the base at right angles.

7. Let ABC be a triangle, in which the angle ABC is a right angle. From AC cut off AD equal to AB, and join BD. Prove that the angle BAC is equal to twice the angle CBD.

8. If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to, &c. (5, II.)

9. In every triangle, the square on the side subtending an acute angle is less than the squares on the sides containing that angle, by &c. (13, II). (It will be sufficent to take the case in which the perpendicular falls within the triangle.)

10. To describe a square that shall be equal to a given rectilineal figure.

11. The square on any straight line drawn from the vertex of an isosceles triangle to the base is less than the square on a side of a triangle by a rectangle contained by the segments of the base.

SECOND CLASS PROVINCIAL CERTIFICATES, 1875.

TIME—TWO HOURS AND THREE-QUARTERS.

NOTE.—Those students who take only Book I. will confine themselves to the first seven questions. Those who take Books I. and II. will omit the questions marked with an asterisk (*), namely, (1) and (2).

*1. If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

*2. If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, the sides opposite to equal angles, then shall the other sides be equal, each to each.

3. If a straight line falling on two other straight lines make the alternate angles equal to each other, these two straight lines shall be parallel.

4. If a straight line fall upon two parallel straight lines, it makes the two interior angles upon the same side together equal to two right angles.
5. Assuming Proposition XXXII, deduce the corollary: "all the exterior angles of any rectilineal figure, made by producing the sides successively in the same direction, are together equal to four right angles."
6. If a straight line, drawn parallel to the base of a triangle, bisect one of the sides, it shall bisect the other also.
7. Let ABC and ADC be two triangles on the same base AC and between the same parallels AC and BD. Prove, that, if the sides AB and BC be equal to one another, their sum is less than the sum of the sides AD and DC.
8. If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.
9. If a straight line be bisected and produced to any point, the rectangles contained by the whole line thus produced, and the part of it produced, together with, etc., (6, II).
10. Divide a straight line into two parts, such that the sum of their squares may be the least possible.

FIRST CLASS PROVINCIAL CERTIFICATES, 1871.

TIME.—THREE HOURS.

1. To describe a square that shall be equal to a given rectilineal figure.
2. A segment of a circle being given, to describe the circle of which it is the segment.
3. If the vertical angle of a triangle be divided into two equal angles by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to one another.
4. In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle and to one another.
5. If four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals.
6. Draw a straight line so as to touch two given circles.
7. Let A B C be a triangle, and from B and C, the extremities of the base B C, let line B F and C E be drawn to F and E, the middle points of A C and A B respect-

ively, then, if B F = C E, A B and A C shall be equal to one another.

8. Describe an equilateral triangle equal to a given triangle.

FIRST CLASS PROVINCIAL CERTIFICATES, 1872.

TIME—TWO AND A HALF HOURS.

1. If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.
2. To inscribe a circle in a given triangle.
3. Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional.
4. Similar triangles are to one another in the duplicate ratio of their homologous sides.
5. In any right angled triangle, any rectilineal figure described on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.
6. Two circles cut each other, and through the points of section are drawn two parallel lines, terminated by the circumferences. Prove that these lines are equal.
7. Let A C and B D, the diagonals of a quadrilateral figure A B C D, intersect in E. Then, if A B be parallel to C D, the circles described about the triangles A B E and C D E shall touch one another.
8. Divide a triangle into two equal parts by a straight line at right angles to one of the sides.

FIRST CLASS PROVINCIAL CERTIFICATES, 1873.

TIME—THREE HOURS.

1. The angle in a semicircle is a right angle.
2. A segment of a circle being given, describe the circle of which it is a segment.
3. Give Euclid's definition of proportion; and prove, by taking equi-multiples according to the definition, that 2, 3, 9, 13, are not proportionals.
4. Similar triangles are to one another in the duplicate ratio of their homologous sides.
5. To find a mean proportional between two given straight lines.
6. Through C, the vertex of a triangle A C B, which has the sides A C and C B equal to one another, a line C D

is drawn parallel to A B; and straight lines, A D, D B, are drawn from A and B to any point D in C D. Prove that the angle A C D is greater than the angle A D B.

7. A B C D is a quadrilateral figure inscribed in a circle. From A and B, perpendiculars A E, B F are let fall on C D (produced if necessary); and from C and D, perpendiculars C G, D H, are let fall on B A (produced if necessary). Prove that the rectangles A E, B F and C G, D H, are equal to one another.

8 A B C D is a quadrilateral figure inscribed in a circle. The straight line D E drawn through D parallel to A B, cuts the side B C in E; and the straight line A E produced meets D C produced in F. Prove, that if the rectangle B A, A D be equal to the rectangle E C, C F, the triangle A D F shall be equal to the quadrilateral A B C D.

FIRST CLASS PROVINCIAL CERTIFICATES, 1874.

TIME—THREE HOURS.

1. In equal circles, equal straight lines cut off equal circumferences, the greater, equal to the greater, and the less to the less.
2. To describe a circle about a given equilateral and equiangular pentagon.
3. To find a mean proportional between two given straight lines.
4. What is meant by duplicate ratio? Write down two whole numbers, which are in the duplicate ratio of $\frac{1}{2}$ to $\frac{1}{3}$.
   What are similar rectilineal figures?
   Similar triangles are to one another in the duplicate ratio of their homologous sides.
5. In any right angled triangle, any rectilineal figure described on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.
6 To describe a triangle, of which the base, the vertical angle, and the sum of the two sides are given.
7. From A the vertex of a triangle ABC, in which each of the angles ABC and ACB is less than right angle, AD is let fall perpendicular on the base BC. Produce BC to E, making CE equal to AD; and let F be a point in AC, such that the triangle BFE is equal to the triangle ABC. Prove that F is one of the angular points of a square inscribed in the triangle ABC, with one of its sides on BC.

8. Let E be the point of intersection of the diagonals of a quadrilateral figure ABCD, of which any two opposite angles are together equal to two right angles. Produce BC to G, making CG equal to EA; and produce AD to F, making DF equal to BE. Prove that if EG and EF be joined, the triangles EDF and ECG are equal to one another.

FIRST CLASS PROVINCIAL CERTIFICATES, 1875.

TIME—THREE HOURS.

1. If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, the sides adjacent to the equal angles in each, then shall the other sides be equal each to each.
2. From a given circle to cut off a segment, which shall contain an angle equal to a given rectilineal angle.
3. If the angle of a triangle be divided into two equal angles by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangles have to one another.
4. The sides about the equal angles equi-angular triangles are proportionals; and those which are opposite to the equal angles are homologous sides.
5. If the similar rectilineal figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.
6. Any rectangle is half the rectangle contained by the diameters of the squares on its adjacent sides.
7. Through a given point within a given circle, to draw a straight line such that one of the parts of it intercepted between that point and the circumference shall be double of the other.
8. If, from any point in a circular arc, perpendiculars be let fall on its bounding radii, the distance of their feet is invariable.

MATRICULATION, 1871.

1. State the points of agreement and disagreement of the circle, square and rhombus, with one another as appearing from their definitions.
2. Any two sides of a triangle are together greater than the third side.

   Show that the sum of the excesses of each pair of sides above the third side is equal to the sum of the three sides of the triangle.

3. If the square described upon one of the sides of a triangle be equal to the square described on the other two sides of it, the angle contained by these two sides is a right angle.

In an isosceles triangle if the square on the base be equal to three times the square on either side the vertical angle is two-thirds of two right angles.

4. If a straight line be divided into any two parts the square on the whole line is equal to the square on the two parts, together with twice the rectangle contained by the parts.

Is there any difference between the principle of this proposition and the statement $(a + b)^2 = a^2 + 2ab + b^2$.

Of all the squares that can be inscribed within another the least is that formed by joining the bisections of the side.

5 If a straight line be divided into two equal and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.

Does the statement respecting the equality of the square hold for any other division of the line.

6. Equal straight lines in a circle are equally distant from the centre; and conversely, those which are equally distant from the centre are equal to one another.

The lines joining the extremities of two equal straight lines in a circle towards the same parts are parallel to each other.

7. What is meant by the Angle in a segment of a circle? Define similar segments of circles.

Upon the same straight line and upon the same side of it, there cannot be two similar segments of circles not coinciding with one another.

8. In equal circles the angles which stand upon equal arcs, are equal to one another whether they be at the centres or circumferences.

If two equal circles so intersect each other that the tangents at one of their points of intersection are inclined to each other at an angle of 60° shew that

Radius of circle : line joining their centres : : 1 : $\sqrt{3}$.

9. From a given circle to cut off a segment that shall contain an angle equal to a given rectilineal angle.

In a given circle inscribe a triangle which shall have a given vertical angle, and whose area shall be equal to a given triangle; and shew with what limitation this can be done.

10. When is a circle said to be inscribed in a rectilinea figure.
To inscribe a circle in a given triangle.
11. Inscribe an equilateral and equiangular pentagon in a given circle.
Show how to divide a right angle into fifteen equal parts.

MATRICULATION, 1872.

HONORS.

1. From a given point to draw a straight line equal to a given straight line.
Explain what different constructions there are in this proposition.
2. If a side of a triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.
Find the number of degrees in one of the exterior angles of a regular heptagon.
3. Triangles upon the same or equal bases and between the same parallels are equal to one another.
By means of these propositions prove that a line drawn parallel to the base of a triangle and cutting off one-fourth from one of its sides, will also cut off a fourth part from the other side.
4. If a straight line be divided into two equal and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line, and of the square on the line between the points of section.
If a chord be drawn parallel to the diameter of a circle and from any point in the diameter lines be drawn to its extremities, the sum of their squares will be equal to the sum of the squares of the segments of the diameter.
5. To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.
Solve the problem algebraically. Interpret and construct geometrically the second root so obtained.
Divide a given line so that one segment may be a geometric mean between the whole and the other.
6. In every triangle, the square on the side subtending either of the acute angles, is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight

line intercepted between the acute angle and the perpendicular let fall upon it from the opposite angle.

In a triangle ABC, if AD be drawn to the bisection of BC, the difference between the square on BC and twice the square on AC is double of the difference between the square on AB, and twice the square on AD

7. If a straight line touch a circle, the straight line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.

The locus of intersections of all pairs of tangents to a circle which contain a given angle is a circle.

What is the magnitude of this angle, in order that the circle may be double the original ?

8. The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

What relation must exist between the sides of a quadrilateral in order that a circle may be inscribed in it? Show that your relation is sufficient.

9. If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square on the line which touches it.

Show that this proposition is an extension of III, 35.

From a given point without a circle show how to draw (when possible) a line that will be divided by that circle in Medial section.

10. Inscribe a circle in a given triangle.

When is one rectilineal figure said to be inscribed in another.

11. In a right-angled triangle, if the perpendicular be drawn from the right angle to the base; the triangle on each side of it are similar to the whole triangle and to one another.

Construct geometrically the roots of the equation $x(a-x) = b^2$ and give the geometric interpretation of the case of equal and impossible roots that the problem may present.

12. To describe a rectilineal figure which shall be similar to one given rectilineal figure and equal to another given rectilineal figure.

MATRICULATION, 1873.

HONORS.

1. If a straight line falls upon two parallel straight lines, it makes the alternate angles equal to one another, and

the exterior angle equal to the interior and opposite upon the same side, and also the two interior angles upon the same side together equal to two right angles.

Vary the order of proof in this proposition by proving the last statement first.

2. If a straight line falling upon two other straight lines, makes the interior angles upon the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Can this be inferred immediately from the 12th axiom? Give the reasons for your answer.

3. Any two sides of a triangle are together greater than the third side.

A straight line is the shortest distance between two given points.

4. In any right angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

Any two parallelograms being described on two sides of any triangle, to describe on the third side a parallelogram equal to their sum.

5. To describe a square that shall equal a given rectilineal figure.

To divide a given straight line into two parts such that their rectangle is equal to a given rectilineal figure.

What limitation must there be to the magnitude of the given figure?

6. If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and, if it cut it at right angles, it shall bisect it.

Describe three circles of given radii which shall touch each other externally two and two.

7. In the above show that the common tangents meet in one point, with which as centre, a circle may be described passing through the three points of contact.

What proposition of Euclid does this correspond to?

8. If straight lines within a circle intersect in one point the rectangle under the segments is constant.

What limitation must be made to render the converse true? Prove the converse when true.

9. The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles. Deduce—The angle in a semicircle is a right angle. (Prop. 31 Bk. III.)

10. To describe an isosceles triangle having each of the angles at the base double of the third angle.

A tangent to a circle is drawn at an angular point of an inscribed regular pentagon, and a side produced through that point, show that a straight line making equal intercepts on the tangent and the side produced, is parallel to the tangent at one of the adjacent angular points.

11. To describe a circle about a given equilateral pentagon.

With an angular point of the regular pentagon as centre, and a side as radius, describe a second circle; show that the tangent to the first circle at a point of intersection of the circles meets the common diameter at a point without the second circle.

12. In the above show that the distance from the above point to the centre of the first circle is greater than the diameter of the second circle.

MATRICULATION, 1874.

HONORS.

⁂ Nos. 1 and 3 to be omitted for Senior Matriculation; Nos. 12 and 13 to be omitted for Junior Matriculation.

1. Parallelograms upon the same base and between the same parallels are equal to one another.

From the centre O of a circle the radii OA, OB are drawn, the tangents at A and B meet in C; if OC be bisected in D and DE be drawn perpendicular to OD meeting OB in E, then AE will bisect the figure $OBCA$.

2. In every triangle the square on the side subtending any of the acute angles is less than the squares on the sides containing that angle by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle and the acute angle.

Construct a square that shall be equal to the difference between the sum of the squares on two given straight lines and the rectangle under these lines.

3. Through a given point to draw a straight line parallel to a given straight line.

From a given point in the circumference of a circle to draw a chord, when possible, that shall be bisected by a given chord.

4. Find the sum of (1) all the interior angles of any rectilineal figure; (2) all the exterior angles.

AB, CD the alternate sides of a regular polygon are produced to meet in E, if AC, OE meet in F, O being the centre of the polygon, show that $AF.FC = OF.FE$

5. To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.

If AB be bisected in C and produced to a point D, such that $AC.CD=AD.DB$, then AD is divided in C in the manner required by the proposition.

6. If from any point without a circle two straight lines be drawn, one of which cuts the circle and the other touches it, the rectangle contained by the whole line that cuts the circle and the part of it without the circle shall be equal to the square on the line that touches it.

Any number of circles pass through two given points A and B; shew that with any given point C in AB produced, as centre, a circle may be described cutting the other circles at right angles, and find its radius.

7. To draw a straight line from a given point either without or in the circumference which shall touch a given circle.

Find the point in the line joining the centres of two circles of different radii, such that if a perpendicular be drawn through it, the tangents to the circles from any point in this perpendicular may be equal.

8. The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.

If a circle be described touching one of the equal sides of an isosceles triangle at the vertex and having the other side as chord, the arc lying between the vertex and base is one-half the arc subtended by the chord.

9. If a straight line touch a given circle and from the point of contact a straight line may be drawn cutting the circle, the angles made by this line with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.

10. To inscribe an equilateral and equiangular pentagon in a given circle.

If two diagonals of a regular pentagon intersect and a circle be described about the triangle of which the greater segments are two sides, two sides of the pentagon which terminate at the other extremities of these segments are tangents to the circle at these points.

11. To describe a circle about a given square.

Find the relation between the areas of the circles described about and inscribed in a given square.

12. If a straight line be parallel to the base of a triangle it will cut the sides, or the sides produced, proportionally, and if the sides, or the sides produced, be cut

proportionally, the straight line which joins the points of section shall be parallel to the base.

13. To find a mean proportional between two given straight lines.

JUNIOR AND SENIOR MATRICULATION, 1875.

*** Junior Matriculants will omit questions 15 and 16, and Senior Matriculants questions 12 and 13.

1. Define the terms axiom, postulate, scholium, corollory.
2. If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of the one greater than the angle contained by the two sides equal to them, of the other, the base of that which has the greater angle shall be greater than the base of the other.
3. If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.
4. Triangles on equal bases and between the same parallels are equal to one another.
5. If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.
6. If the diagonals of a quadrilateral bisect each other, it is a parallelogram: if the bisecting lines are equal it is rectangular; if the lines bisect at right angles it is equilateral.
7. If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.
8. Divide a straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.
9. In the Algebraic solution of the preceding problem, we obtain a quadratic equation which gives two values of the unknown quantity. Enunciate the Geometrical proposition which corresponds to the other root.
10. The sum of the squares on the diagonals of a parallelogram is equal to the sum of the squares on the sides.
11. The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.
12. The straight lines bisecting the sides of a triangle at right angles meet in a point.

13. Construct a triangle, having given the middle points of sides.
14. Describe a circle about a given equilateral and equiangular pentagon.
15. From a given straight line to cut off any part required.
16. Similar triangles are to one another in the duplicate ratio of their homologous sides.

TIME—3 HOURS.

1. Describe an equilateral triangle upon a given finite straight line.

   By a method similar to that used in this problem, describe on a given finite straight line an isosceles triangle, the sides of which shall be each equal to twice the base.
2. If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

   What objections have been urged against the doctrine of parallel straight lines as it is laid down by Euclid? Where does the difficulty originate and what has been suggested to remove it?
3. In any right angled triangle, the squares described on the sides containing the right angle are together equal to the square of the side subtending the right angle.

   Show, by describing a square on the outer side of one side, and on the inner side of the other, that the two squares thus described will cut into *three* pieces, so as exactly to make up the square of the hypotenuse.
4. Divide *algebraically* a given line (a) into two parts, such that the rectangle contained by the whole and one part may be equal to the square of the other. Deduce Euclid's construction from one solution and explain the other.
5. If two straight lines within a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

   If, through a point within a circle, two equal straight lines be drawn to the circumference, and produced, they will be at the same distance from the centre.
6. Explain and illustrate the fifth and seventh definitions in the fifth book of Euclid, and shew that a magnitude has a greater ratio to the less of two unequal magnitudes than it has to the greater.
7. With the four lines contain $a+b$, $a+c$, $a-b$, $a-c$ units respectively, construct a quadrilateral capable of having a circle inscribed in it.

Prove that no parallelogram can be inscribed in a circle except a rectangle; and that no parallelogram can be described about a circle except a rhomb.

8. Similar triangles are to one another in the duplicate ratio of their homologous sides. How does it appear from Euclid that the duplicate ratio of two magnitudes is the same as that of their squares?

FIRST CLASS PROVINCIAL CERTIFICATES, JULY, 1876.

TIME—THREE HOURS.

N. B.—Algebraic symbols must not be used.

1. (*a*) The straight line drawn at right angles to the diameter of a circle from the extremity of it, falls without the circle; and no straight line can be drawn from the extremity, between that straight line and the circumference, so as not to cut the circle. (III 16.)

   (*b*) Draw a common tangent to two given circles. How many can be drawn? (*Apollonius.*)

2. (*a*) The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles. (III 22.)

   (*b*) If straight lines be drawn from any point on the circumference of a circle perpendicular to the sides of an inscribed triangle, their feet are in the same straight line. (*M. F. Jacobi.*)

3. (*a*) If the chord of a circle be divided into two segments by a point in the chord or in the chord produced, the rectangle contained by these segments will be equal to the difference of the squares on the radius and on the line joining the given point within the centre of the circle. What propositions in Euclid follow immediately from this?

   (*b*) Describe a circle which shall pass through a given point and touch two straight lines given in position. (*Apollonius.*)

4. (*a*) To describe an isosceles triangle, having each of the angles at the base double of the third angle. (IV 10.)

   (*b*) Construct a triangle having each of the angles at the base equal to seven times the third angle.

5. (*a*) If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base have the same ratio which the other sides of the triangle have to one another; and, if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line

drawn from the vertex to the point of section shall bisect the vertical angle. (VI 3.)

(b) The points in which the bisectors of the external angles of a triangle meet the opposite sides, lie in a straight line.

SECOND CLASS CERTIFICATES, JULY, 1876.

TIME—THREE HOURS.

N. B.—Algebraic symbols must not be used. Candidates who take Book II will omit Questions 1, 2 *and* 3, *marked* *.

Values.

16 *1. The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles on the other side of the base shall be equal to one another.

3 Where does Euclid require the second part of this theorum?

16 *2. If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by two sides of one of them greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle shall be greater than the base of the other.

6 Why the restriction "Of the two sides DE, DF, let DE be the side which is not greater than the other"?

16 *3. If two triangles have two angles of the one equal to two angles of the other, each to each, and have also the sides adjacent to the equal angles in each, equal to one another, then shall the other side be equal, each to each; and also the third angle of the one to the third angle of the other. (Prove by superposition.)

3 What propositions in Book I are thus proved?

16 4. If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

8 What objection may be taken to the twelfth axiom?

2 What is its converse?

16 5. In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

Prove also by dissection and superposition.

Draw through a given point between two straight lines not parallel a straight line which shall be bisected in that point.

The perpendiculars from the angles of a triangle on the opposite sides meet in a point.

20 8. Given the lengths of the lines drawn from the angles of a triangle to the points of bisection of the opposite sides, construct the triangle.

20 9. If a straight line be divided into two parts, the square on the whole line is equal to the squares on the parts, together with twice the rectangle contained by the parts.

20 10. In every triangle, the square on the side subtending an acute angle is less than the squares on the sides containing that angle by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle, and the acute angle.